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$\mathcal{W}$-extended fusion algebra of critical percolation

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Abstract

Two-dimensional critical percolation is the member $\mathcal{LM}(2, 3)$ of the infinite series of Yang–Baxter integrable logarithmic minimal models $\mathcal{LM}(p, p')$. We consider the continuum scaling limit of this lattice model as a 'rational' logarithmic conformal field theory with extended $\mathcal{W} = \mathcal{W}_{2,3}$ symmetry and use a lattice approach on a strip to study the fundamental fusion rules in this extended picture. We find that the representation content of the ensuing closed fusion algebra contains 26 $\mathcal{W}$-indecomposable representations with eight rank-1 representations, 14 rank-2 representations and four rank-3 representations. We identify these representations with suitable limits of Yang–Baxter integrable boundary conditions on the lattice and obtain their associated $\mathcal{W}$-extended characters. The latter decompose as finite non-negative sums of $\mathcal{W}$-irreducible characters of which 13 are required. Implementation of fusion on the lattice allows us to read off the fusion rules governing the fusion algebra of the 26 representations and to construct an explicit Cayley table. The closure of these representations among themselves under fusion is remarkable confirmation of the proposed extended symmetry.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The study of percolation [1–4] as a lattice model has a long history [5–7]. In this paper, it is convenient to regard two-dimensional critical percolation as the member $\mathcal{LM}(2, 3)$ of the infinite series of Yang–Baxter integrable logarithmic minimal models $\mathcal{LM}(p, p')$ [8]. It is a well-established principle that two-dimensional lattice systems in general [9] and percolation in particular [10, 11] are conformally invariant in the continuum scaling limit. Our lattice approach to studying these conformal field theories is predicated on the supposition that, in the continuum scaling limit, a transfer matrix with prescribed boundary conditions gives rise to a representation of the Virasoro algebra. Different boundary conditions naturally lead to different representations which can be of different types—reducible or irreducible, decomposable or indecomposable. We further assume that, if in addition, the boundary conditions respect the symmetry of a larger conformal algebra $\mathcal{W}$, then the continuum scaling limit of the transfer matrix will yield a representation of the extended algebra $\mathcal{W}$.

Notwithstanding the fact that critical percolation is one of the very few systems which has been rigorously shown [12] to be conformally invariant in the continuum scaling limit, the study of critical percolation as a conformal field theory (CFT) is not so well advanced. In large part, this is because critical percolation [13–19], like critical dense polymers $\mathcal{LM}(1, 2)$ [20–24] or symplectic fermions [25, 26], is a prototypical *logarithmic* CFT. The properties [27–29] of logarithmic CFTs differ dramatically from the familiar properties of *rational* CFTs. In particular, they are non-rational and non-unitary with a countably infinite number of scaling fields. Unlike rational CFTs, whose field or representation content consists entirely of *irreducible* Virasoro representations, logarithmic CFTs admit *reducible yet indecomposable* representations [30] of the Virasoro algebra. These representations, some of which are accompanied by non-trivial Jordan-cell structures for the Virasoro dilatation generator L_0, play an essential role and are in fact characteristic of logarithmic CFTs.

Recently, Virasoro fusion rules have been proposed [31–35] for all the augmented minimal or logarithmic minimal models $\mathcal{LM}(p, p')$. Interestingly, it was found that only indecomposable representations of rank 1, 2 or 3 appear corresponding to Jordan cells of dimension 1, 2 or 3, respectively. However, a central question of much current interest [36–39] is whether an extended symmetry algebra $\mathcal{W}$ exists for these logarithmic theories. Such a symmetry should allow the countably *infinite* number of Virasoro representations to be reorganized into a *finite* number of extended $\mathcal{W}$-representations which close under fusion. In the case of the logarithmic minimal models $\mathcal{LM}(1, p)$, the existence of such an extended $\mathcal{W}$-symmetry and the associated fusion rules are by now well established [37, 40–44]. By stark contrast, although there are strong indications [45, 46] that there exists a $\mathcal{W}_{p,p'}$ symmetry algebra for general augmented minimal models, very little is known about the $\mathcal{W}$-extended fusion rules for the $\mathcal{LM}(p, p')$ models with $p \geqslant 2$.

In this paper, we use a lattice approach on a strip, generalizing the approach of [44], to obtain fusion rules of critical percolation $\mathcal{LM}(2, 3)$ in the extended symmetry picture. In [44], it was shown that in fact symplectic fermions is just critical dense polymers $\mathcal{LM}(1, 2)$ viewed in the extended picture. Likewise in the case of critical percolation, the extended picture is described by the *same* lattice model as the Virasoro picture. We nevertheless find it useful to distinguish between the two pictures by denoting the extended picture $\mathcal{WLM}(2, 3)$ and thus reserve the notation $\mathcal{LM}(2, 3)$ for critical percolation in the non-extended Virasoro picture. A similar distinction applies to the entire infinite series of logarithmic minimal models. These $\mathcal{W}$-extended models, which we denote by $\mathcal{WLM}(p, p')$, are discussed in [47]. The $\mathcal{W}$-extended fusion rules we obtain for critical percolation are based on the *fundamental* fusion algebra in the Virasoro picture [34, 35] which is a subset of the *full* fusion algebra. The latter remains to be determined and may eventually yield a larger $\mathcal{W}$-extended fusion algebra than the one presented here.

The layout of this paper is as follows. In section 2, we review the Virasoro fusion rules for critical percolation [34]. In section 3, we summarize the $\mathcal{W}$-representation content consisting of 26 $\mathcal{W}$-indecomposable representations with eight rank-1 representations, 14 rank-2 representations and four rank-3 representations and present their associated extended characters. The latter decompose as finite non-negative sums of $\mathcal{W}$-irreducible characters of which 13 are required. These are all identified. Lastly, in this section, we present the explicit Cayley table of the fundamental $\mathcal{W}$-extended fusion rules obtained by implementing fusion on the lattice. In section 4, we identify the $\mathcal{W}$-extended representations with suitable limits of Yang–Baxter integrable boundary conditions on the lattice and give details of their construction and properties. We conclude with a short discussion. Throughout, we use the notation $\mathbb{Z}_{n,m} = \mathbb{Z} \cap [n, m]$, with $n, m \in \mathbb{Z}$, to denote the set of integers from n to m, both included, and denote an n-fold fusion of the representation A with itself by

$$A^{\otimes n} = \underbrace{A \otimes A \otimes \cdots \otimes A}_{n}. \tag{1.1}$$

2. Critical percolation $\mathcal{LM}(2,3)$

2.1. Logarithmic minimal model $\mathcal{LM}(p, p')$

A logarithmic minimal model $\mathcal{LM}(p, p')$ is defined [8] for every coprime pair of positive integers $p < p'$. The model $\mathcal{LM}(p, p')$ has central charge

$$c = 1 - 6\frac{(p'-p)^2}{pp'} \tag{2.1}$$

and conformal weights

$$\Delta_{r,s} = \frac{(rp'-sp)^2 - (p'-p)^2}{4pp'}, \qquad r, s \in \mathbb{N}. \tag{2.2}$$

The fundamental fusion algebra $\langle(2, 1), (1, 2)\rangle_{p,p'}$ [34, 35] of the logarithmic minimal model $\mathcal{LM}(p, p')$ is generated by the two fundamental Kac representations $(2, 1)$ and $(1, 2)$ and contains a countably infinite number of inequivalent, indecomposable representations of rank 1, 2 or 3. For $r, s \in \mathbb{N}$, the character of the Kac representation (r, s) is

$$\chi_{r,s}(q) = \frac{q^{\frac{1-c}{24}+\Delta_{r,s}}}{\eta(q)}(1 - q^{rs}) = \frac{1}{\eta(q)}\left(q^{(rp'-sp)^2/4pp'} - q^{(rp'+sp)^2/4pp'}\right), \tag{2.3}$$

where the Dedekind eta function is given by

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n). \tag{2.4}$$

Such a representation is of rank 1 and is irreducible if $r \in \mathbb{Z}_{1,p}$ and $s \in p'\mathbb{N}$ or if $r \in p\mathbb{N}$ and $s \in \mathbb{Z}_{1,p'}$. It is a reducible yet indecomposable representation if $r \in \mathbb{Z}_{1,p-1}$ and $s \in \mathbb{Z}_{1,p'-1}$, while it is a fully reducible representation if $r \in p\mathbb{N}$ and $s \in p'\mathbb{N}$, where

$$(kp, k'p') = (k'p, kp') = \bigoplus_{j=|k-k'|+1,\ \text{by}\ 2}^{k+k'-1} (jp, p') = \bigoplus_{j=|k-k'|+1,\ \text{by}\ 2}^{k+k'-1} (p, jp'). \tag{2.5}$$

These are the only Kac representations appearing in the fundamental fusion algebra. The characters of the reducible yet indecomposable Kac representations just mentioned can be written as sums of two irreducible Virasoro characters

$$\chi_{r,s}(q) = \mathrm{ch}_{r,s}(q) + \mathrm{ch}_{2p-r,s}(q) = \mathrm{ch}_{r,s}(q) + \mathrm{ch}_{r,2p'-s}(q), \qquad r \in \mathbb{Z}_{1,p-1}, \qquad s \in \mathbb{Z}_{1,p'-1}. \tag{2.6}$$

In general and with $r_0 \in \mathbb{Z}_{1,p-1}$, $s_0 \in \mathbb{Z}_{1,p'-1}$ and $k \in \mathbb{N}-1$, the irreducible Virasoro characters read [48]

$$\mathrm{ch}_{r_0+kp,s_0}(q) = K_{2pp',(r_0+kp)p'-s_0p;k}(q) - K_{2pp',(r_0+kp)p'+s_0p;k}(q)$$
$$\mathrm{ch}_{r_0+(k+1)p,p'}(q) = \frac{1}{\eta(q)}\left(q^{(kp+r_0)^2p'/4p} - q^{((k+2)p-r_0)^2p'/4p}\right)$$

$$\mathrm{ch}_{(k+1)p,s_0}(q) = \frac{1}{\eta(q)}\big(q^{((k+1)p'-s_0)^2p/4p'} - q^{((k+1)p'+s_0)^2p/4p'}\big)$$
$$\mathrm{ch}_{(k+1)p,p'}(q) = \frac{1}{\eta(q)}\big(q^{k^2pp'/4} - q^{(k+2)^2pp'/4}\big), \tag{2.7}$$

where $K_{n,\nu;k}(q)$ is defined as

$$K_{n,\nu;k}(q) = \frac{1}{\eta(q)}\sum_{j\in\mathbb{Z}\setminus\mathbb{Z}_{1,k}} q^{(\nu-jn)^2/2n}. \tag{2.8}$$

For $r\in\mathbb{Z}_{1,p}$, $s\in\mathbb{Z}_{1,p'}$, $a\in\mathbb{Z}_{1,p-1}$, $b\in\mathbb{Z}_{1,p'-1}$ and $k\in\mathbb{N}$, the representations denoted by $\mathcal{R}_{kp,s}^{a,0}$ and $\mathcal{R}_{r,kp'}^{0,b}$ are indecomposable representations of rank 2, while $\mathcal{R}_{kp,p'}^{a,b}\equiv\mathcal{R}_{p,kp'}^{a,b}$ is an indecomposable representation of rank 3. Their characters read

$$\begin{aligned}
\chi\big[\mathcal{R}_{kp,s}^{a,0}\big](q) &= (1-\delta_{k,1}\delta_{s,p'})\mathrm{ch}_{kp-a,s}(q) + 2\mathrm{ch}_{kp+a,s}(q) + \mathrm{ch}_{(k+2)p-a,s}(q)\\
\chi\big[\mathcal{R}_{r,kp'}^{0,b}\big](q) &= (1-\delta_{k,1}\delta_{r,p})\mathrm{ch}_{r,kp'-b}(q) + 2\mathrm{ch}_{r,kp'+b}(q) + \mathrm{ch}_{r,(k+2)p'-b}(q)\\
\chi\big[\mathcal{R}_{kp,p'}^{a,b}\big](q) &= (1-\delta_{k,1})\mathrm{ch}_{(k-1)p-a,b}(q) + 2\mathrm{ch}_{(k-1)p+a,b}(q) + 2(1-\delta_{k,1})\mathrm{ch}_{kp-a,p'-b}(q)\\
&\quad + 4\mathrm{ch}_{kp+a,p'-b}(q) + (2-\delta_{k,1})\mathrm{ch}_{(k+1)p-a,b}(q) + 2\mathrm{ch}_{(k+1)p+a,b}(q)\\
&\quad + 2\mathrm{ch}_{(k+2)p-a,p'-b}(q) + \mathrm{ch}_{(k+3)p-a,b}(q)\\
&= (1-\delta_{k,1})\mathrm{ch}_{a,(k-1)p'-b}(q) + 2\mathrm{ch}_{a,(k-1)p'+b}(q) + 2(1-\delta_{k,1})\mathrm{ch}_{p-a,kp'-b}(q)\\
&\quad + 4\mathrm{ch}_{p-a,kp'+b}(q) + (2-\delta_{k,1})\mathrm{ch}_{a,(k+1)p'-b}(q) + 2\mathrm{ch}_{a,(k+1)p'+b}(q)\\
&\quad + 2\mathrm{ch}_{p-a,(k+2)p'-b}(q) + \mathrm{ch}_{a,(k+3)p'-b}(q).
\end{aligned} \tag{2.9}$$

For $a\in\mathbb{Z}_{0,p-1}$, $b\in\mathbb{Z}_{0,p'-1}$ and $k,k'\in\mathbb{N}$, a decomposition similar to (2.5) applies to the higher-rank *decomposable* representations $\mathcal{R}_{kp,k'p'}^{a,b}$ as we have

$$\mathcal{R}_{kp,k'p'}^{a,b} = \mathcal{R}_{k'p,kp'}^{a,b} = \bigoplus_{j=|k-k'|+1,\ \mathrm{by}\ 2}^{k+k'-1}\mathcal{R}_{jp,p'}^{a,b} = \bigoplus_{j=|k-k'|+1,\ \mathrm{by}\ 2}^{k+k'-1}\mathcal{R}_{p,jp'}^{a,b}. \tag{2.10}$$

Here we have introduced the convenient notation

$$\mathcal{R}_{r,s}^{0,0}\equiv(r,s). \tag{2.11}$$

Fusion in the fundamental fusion algebra $\langle(2,1),(1,2)\rangle_{p,p'}$ decomposes into 'horizontal' and 'vertical' components. With $a\in\mathbb{Z}_{0,p-1}$, $b\in\mathbb{Z}_{0,p'-1}$ and $k\in\mathbb{N}$, we thus have

$$\mathcal{R}_{p,kp'}^{a,b} = \mathcal{R}_{p,1}^{a,0}\otimes\mathcal{R}_{1,kp'}^{0,b} = \mathcal{R}_{kp,1}^{a,0}\otimes\mathcal{R}_{1,p'}^{0,b}. \tag{2.12}$$

The Kac representation $(1,1)$ is the identity of the fundamental fusion algebra. For $p>1$, this is a reducible yet indecomposable representation, while for $p=1$, it is an irreducible representation. Below, we summarize the fusion rules in the case of critical percolation $\mathcal{LM}(2,3)$. The associated extended Kac table is given in figure 1.

2.2. *Fundamental fusion algebra of* $\mathcal{LM}(2,3)$

The fundamental fusion algebra $\langle(2,1),(1,2)\rangle_{2,3}$ is generated by the irreducible Kac representation $(2,1)$ and the reducible yet indecomposable Kac representation $(1,2)$ and contains a variety of representations

$$\langle(2,1),(1,2)\rangle_{2,3} = \big\langle(1,1),(1,2),(2k,s),(r,3k),\mathcal{R}_{2k,s}^{1,0},\mathcal{R}_{r,3k}^{0,b},\mathcal{R}_{2k,3}^{1,b}\big\rangle_{2,3}, \tag{2.13}$$

where $r,b\in\mathbb{Z}_{1,2}$, $s\in\mathbb{Z}_{1,3}$ and $k\in\mathbb{N}$. The representations $(2k,3)\equiv(2,3k)$ are listed twice and it is recalled that $\mathcal{R}_{2k,3}^{1,0}\equiv\mathcal{R}_{2,3k}^{1,0}$, $\mathcal{R}_{2k,3}^{0,b}\equiv\mathcal{R}_{2,3k}^{0,b}$ and $\mathcal{R}_{2k,3}^{1,b}\equiv\mathcal{R}_{2,3k}^{1,b}$. As already mentioned,

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Figure 1. Extended Kac table of critical percolation $\mathcal{LM}(2, 3)$ showing the conformal weights $\Delta_{r,s}$ of the Kac representations (r, s) where $r, s \in \mathbb{N}$. Except for the identifications $(2k, 3k') = (2k', 3k)$, the entries relate to *distinct* Kac representations even if the conformal weights coincide. This is unlike the irreducible representations which are uniquely characterized by their conformal weight. The Kac representations which happen to be irreducible representations are marked with a red-shaded quadrant in the top-right corner. These do not exhaust the distinct values of the conformal weights. For example, the irreducible representation with $\Delta_{1,1} = 0$ does not arise as a Kac representation. By contrast, the Kac table of the associated *rational* (minimal) model consisting of the shaded 1×2 grid in the lower-left corner is trivial and contains only the operator corresponding to the irreducible representation with $\Delta = 0$.

the reducible yet indecomposable Kac representation (1, 1) is the identity of the fundamental fusion algebra

$$(1, 1) \otimes A = A, \tag{2.14}$$

where A is any of the representations listed in (2.13). Thanks to the decomposition illustrated in (2.12), the fundamental fusion algebra follows from a straightforward merge of the horizontal and vertical components. To appreciate this, we follow [34] and let $A_{r,s} = \bar{a}_{r,1} \otimes a_{1,s}$, $B_{r',s'} = \bar{b}_{r',1} \otimes b_{1,s'}$, $\bar{a}_{r,1} \otimes \bar{b}_{r',1} = \bigoplus_{r''} \bar{c}_{r'',1}$ and $a_{1,s} \otimes b_{1,s'} = \bigoplus_{s''} c_{1,s''}$. Our fusion prescription now yields

$$A_{r,s} \otimes B_{r',s'} = (\bar{a}_{r,1} \otimes a_{1,s}) \otimes (\bar{b}_{r',1} \otimes b_{1,s'}) = (\bar{a}_{r,1} \otimes \bar{b}_{r',1}) \otimes (a_{1,s} \otimes b_{1,s'})$$
$$= \Big(\bigoplus_{r''} \bar{c}_{r'',1}\Big) \otimes \Big(\bigoplus_{s''} c_{1,s''}\Big) = \bigoplus_{r'',s''} C_{r'',s''}, \qquad (2.15)$$

where $C_{r'',s''} = \bar{c}_{r'',1} \otimes c_{1,s''}$. In order to describe the component fusion algebras explicitly, we introduce the Kronecker delta combinations [34]

$$\delta^{(2)}_{j,\{k,k'\}} = 2 - \delta_{j,|k-k'|} - \delta_{j,k+k'}$$
$$\delta^{(4)}_{j,\{k,k'\}} = 4 - 3\delta_{j,|k-k'|-1} - 2\delta_{j,|k-k'|} - \delta_{j,|k-k'|+1} - \delta_{j,k+k'-1} - 2\delta_{j,k+k'} - 3\delta_{j,k+k'+1}, \qquad (2.16)$$

where $k, k' \in \mathbb{N}$. The horizontal fusion algebra

$$\langle (2,1)\rangle_{2,3} = \big\langle (2k,1), \mathcal{R}^{1,0}_{2k,1}\big\rangle_{2,3} \qquad (2.17)$$

then reads

$$(2k,1) \otimes (2k',1) = \bigoplus_{j=|k-k'|+1,\ \mathrm{by}\ 2}^{k+k'-1} \mathcal{R}^{1,0}_{2j,1}$$
$$(2k,1) \otimes \mathcal{R}^{1,0}_{2k',1} = \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} (2j,1) \qquad (2.18)$$
$$\mathcal{R}^{1,0}_{2k,1} \otimes \mathcal{R}^{1,0}_{2k',1} = \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} \mathcal{R}^{1,0}_{2j,1},$$

while the vertical fusion algebra

$$\langle (1,2)\rangle_{2,3} = \big\langle (1,1), (1,2), (1,3k), \mathcal{R}^{0,1}_{1,3k}, \mathcal{R}^{0,2}_{1,3k}\big\rangle_{2,3} \qquad (2.19)$$

reads

$$(1,1) \otimes A = A$$
$$(1,2) \otimes (1,2) = (1,1) \oplus (1,3)$$
$$(1,2) \otimes (1,3k) = \mathcal{R}^{0,1}_{1,3k}$$
$$(1,2) \otimes \mathcal{R}^{0,1}_{1,3k} = \mathcal{R}^{0,2}_{1,3k} \oplus 2(1,3k)$$
$$(1,2) \otimes \mathcal{R}^{0,2}_{1,3k} = \mathcal{R}^{0,1}_{1,3k} \oplus (1,3(k-1)) \oplus (1,3(k+1))$$
$$(1,3k) \otimes (1,3k') = \bigoplus_{j=|k-k'|+1,\ \mathrm{by}\ 2}^{k+k'-1} \big(\mathcal{R}^{0,2}_{1,3j} \oplus (1,3j)\big)$$
$$(1,3k) \otimes \mathcal{R}^{0,1}_{1,3k'} = \Bigg(\bigoplus_{j=|k-k'|+1,\ \mathrm{by}\ 2}^{k+k'-1} 2\mathcal{R}^{0,1}_{1,3j}\Bigg) \oplus \Bigg(\bigoplus_{j=|k-k'|,\ \mathrm{by}\ 2}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} (1,3j)\Bigg)$$
$$(1,3k) \otimes \mathcal{R}^{0,2}_{1,3k'} = \Bigg(\bigoplus_{j=|k-k'|,\ \mathrm{by}\ 2}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} \mathcal{R}^{0,1}_{1,3j}\Bigg) \oplus \Bigg(\bigoplus_{j=|k-k'|+1,\ \mathrm{by}\ 2}^{k+k'-1} 2(1,3j)\Bigg)$$
$$\mathcal{R}^{0,1}_{1,3k} \otimes \mathcal{R}^{0,1}_{1,3k'} = \Bigg(\bigoplus_{j=|k-k'|,\ \mathrm{by}\ 2}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} \mathcal{R}^{0,1}_{1,3j}\Bigg) \oplus \Bigg(\bigoplus_{j=|k-k'|+1,\ \mathrm{by}\ 2}^{k+k'-1} \big(2\mathcal{R}^{0,2}_{1,3j} \oplus 4(1,3j)\big)\Bigg)$$
$$\mathcal{R}^{0,1}_{1,3k} \otimes \mathcal{R}^{0,2}_{1,3k'} = \Bigg(\bigoplus_{j=|k-k'|+1,\ \mathrm{by}\ 2}^{k+k'-1} 2\mathcal{R}^{0,1}_{1,3j}\Bigg) \oplus \Bigg(\bigoplus_{j=|k-k'|,\ \mathrm{by}\ 2}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} \big(\mathcal{R}^{0,2}_{1,3j} \oplus 2(1,3j)\big)\Bigg)$$

$$\mathcal{R}_{1,3k}^{0,2} \otimes R_{1,3k'}^{0,2} = \left(\bigoplus_{j=|k-k'|,\text{ by }2}^{k+k'} \delta_{j,\{k,k'\}}^{(2)} \mathcal{R}_{1,3j}^{0,1} \right) \oplus \left(\bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} 2\mathcal{R}_{1,3j}^{0,2} \right) \oplus \left(\bigoplus_{j=|k-k'|-1,\text{ by }2}^{k+k'+1} \delta_{j,\{k,k'\}}^{(4)} (1,3j) \right), \tag{2.20}$$

where A is any of the representations listed in (2.19). To illustrate the merge of the two components, we conclude this discussion of critical percolation in the Virasoro picture $\mathcal{LM}(2,3)$ by considering the fusion

$$\begin{aligned}\mathcal{R}_{2k,3}^{1,1} \otimes \mathcal{R}_{2k',3}^{1,1} &= \big(\mathcal{R}_{2k,1}^{1,0} \otimes \mathcal{R}_{1,3}^{0,1}\big) \otimes \big(\mathcal{R}_{2k',1}^{1,0} \otimes \mathcal{R}_{1,3}^{0,1}\big) = \big(\mathcal{R}_{2k,1}^{1,0} \otimes \mathcal{R}_{2k',1}^{1,0}\big) \otimes \big(\mathcal{R}_{1,3}^{0,1} \otimes \mathcal{R}_{1,3}^{0,1}\big) \\ &= \left(\bigoplus_{j=|k-k'|}^{k+k'} \delta_{j,\{k,k'\}}^{(2)} \mathcal{R}_{2j,1}^{1,0} \right) \otimes \big(\mathcal{R}_{1,6}^{0,1} \oplus 2\mathcal{R}_{1,3}^{0,2} \oplus 4(1,3)\big) \\ &= \left(\bigoplus_{j=|k-k'|-1}^{k+k'+1} \delta_{j,\{k,k'\}}^{(4)} \mathcal{R}_{2j,3}^{1,1} \right) \oplus \left(\bigoplus_{j=|k-k'|}^{k+k'} \delta_{j,\{k,k'\}}^{(2)} \big(2\mathcal{R}_{2j,3}^{1,2} \oplus 4\mathcal{R}_{2j,3}^{1,0}\big) \right). \end{aligned} \tag{2.21}$$

3. $\mathcal{W}$-Extended critical percolation $\mathcal{WLM}(2,3)$

In this section, we summarize our findings in the extended picture for the representation content, their characters and their closed fusion algebra. Unless otherwise specified, we let $\kappa, r, b \in \mathbb{Z}_{1,2}$, $s \in \mathbb{Z}_{1,3}$ and $k, k' \in \mathbb{N}$ in the following.

3.1. Summary of representation content

We have the eight $\mathcal{W}$-indecomposable rank-1 representations

$$\{(2\kappa, s)_{\mathcal{W}}, (r, 3\kappa)_{\mathcal{W}}\} \qquad \text{subject to} \quad (2,6)_{\mathcal{W}} \equiv (4,3)_{\mathcal{W}}, \tag{3.1}$$

where $(2,3)_{\mathcal{W}}$ is listed twice, the 14 $\mathcal{W}$-indecomposable rank-2 representations

$$\big\{\big(\mathcal{R}_{2\kappa,s}^{1,0}\big)_{\mathcal{W}}, \big(\mathcal{R}_{r,3\kappa}^{0,b}\big)_{\mathcal{W}}\big\} \tag{3.2}$$

and the four $\mathcal{W}$-indecomposable rank-3 representations

$$\big\{\big(\mathcal{R}_{2,3}^{1,b}\big)_{\mathcal{W}}, \big(\mathcal{R}_{2,6}^{1,b}\big)_{\mathcal{W}}, \big(\mathcal{R}_{4,3}^{1,b}\big)_{\mathcal{W}}\big\} \qquad \text{subject to} \quad \big(\mathcal{R}_{2,6}^{1,b}\big)_{\mathcal{W}} \equiv \big(\mathcal{R}_{4,3}^{1,b}\big)_{\mathcal{W}}. \tag{3.3}$$

Here we are asserting that these $\mathcal{W}$-representations are indeed $\mathcal{W}$-indecomposable. In terms of Virasoro-indecomposable representations, the $\mathcal{W}$-indecomposable rank-1 representations decompose as

$$\begin{aligned} (2\kappa, s)_{\mathcal{W}} &= \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)(2(2k-2+\kappa), s) \\ (r, 3\kappa)_{\mathcal{W}} &= \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)(r, 3(2k-2+\kappa)), \end{aligned} \tag{3.4}$$

where the two expressions for $(2,3)_{\mathcal{W}}$ agree and where

$$(2,6)_{\mathcal{W}} \equiv (4,3)_{\mathcal{W}}. \tag{3.5}$$

Likewise, the $\mathcal{W}$-indecomposable rank-2 representations decompose as

$$\begin{aligned} \big(\mathcal{R}_{2\kappa,s}^{1,0}\big)_{\mathcal{W}} &= \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa) \mathcal{R}_{2(2k-2+\kappa),s}^{1,0} \\ \big(\mathcal{R}_{r,3\kappa}^{0,b}\big)_{\mathcal{W}} &= \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa) \mathcal{R}_{r,3(2k-2+\kappa)}^{0,b}. \end{aligned} \tag{3.6}$$

Finally, the $\mathcal{W}$-indecomposable rank-3 representations decompose as

$$\begin{aligned}\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}} &= \bigoplus_{k\in\mathbb{N}}(2k-2+\kappa)\mathcal{R}_{2(2k-2+\kappa),3}^{1,b}\\ \big(\mathcal{R}_{2,3\kappa}^{1,b}\big)_{\mathcal{W}} &= \bigoplus_{k\in\mathbb{N}}(2k-2+\kappa)\mathcal{R}_{2,3(2k-2+\kappa)}^{1,b},\end{aligned} \tag{3.7}$$

where the two expressions for $\big(\mathcal{R}_{2,3}^{1,b}\big)_{\mathcal{W}}$ agree and where

$$\big(\mathcal{R}_{2,6}^{1,b}\big)_{\mathcal{W}} \equiv \big(\mathcal{R}_{4,3}^{1,b}\big)_{\mathcal{W}}. \tag{3.8}$$

3.2. *Summary of $\mathcal{W}$-extended characters*

The characters of the $\mathcal{W}$-indecomposable rank-1 representations read

$$\begin{aligned}\hat{\chi}_{2\kappa,s}(q) &= \sum_{k\in\mathbb{N}}(2k-2+\kappa)\mathrm{ch}_{2(2k-2+\kappa),s}(q),\\ \hat{\chi}_{r,3\kappa}(q) &= \sum_{k\in\mathbb{N}}(2k-2+\kappa)\mathrm{ch}_{r,3(2k-2+\kappa)}(q),\end{aligned} \tag{3.9}$$

where it is recalled that $(4,3)_{\mathcal{W}} \equiv (2,6)_{\mathcal{W}}$. The characters of the $\mathcal{W}$-indecomposable rank-2 representations read

$$\begin{aligned}\chi\big[\big(\mathcal{R}_{2\kappa,s}^{1,0}\big)_{\mathcal{W}}\big](q) &= \delta_{\kappa,1}\{1-\delta_{s,3}\} + \sum_{k\in\mathbb{N}}4k\mathrm{ch}_{4k+1,s}(q) + \sum_{k\in\mathbb{N}}(4k-2)\mathrm{ch}_{4k-1,s}(q)\\ \chi\big[\big(\mathcal{R}_{r,3\kappa}^{0,b}\big)_{\mathcal{W}}\big](q) &= \delta_{\kappa,1}\{1-\delta_{r,2}\} + \sum_{k\in\mathbb{N}}(4k+2-2\kappa)\mathrm{ch}_{r,6k+6-3\kappa-b}(q)\\ &\quad + \sum_{k\in\mathbb{N}}(4k-4+2\kappa)\mathrm{ch}_{r,6k-6+3\kappa+b}(q).\end{aligned} \tag{3.10}$$

We note the character identities

$$\chi\big[\big(\mathcal{R}_{2,3}^{1,0}\big)_{\mathcal{W}}\big](q) = \chi\big[\big(\mathcal{R}_{4,3}^{1,0}\big)_{\mathcal{W}}\big](q), \qquad \chi\big[\big(\mathcal{R}_{2,3}^{0,b}\big)_{\mathcal{W}}\big](q) = \chi\big[\big(\mathcal{R}_{2,6}^{0,3-b}\big)_{\mathcal{W}}\big](q) \tag{3.11}$$

and the character relations

$$\chi\big[\big(\mathcal{R}_{2,b}^{1,0}\big)_{\mathcal{W}}\big](q) = 1 + \chi\big[\big(\mathcal{R}_{4,b}^{1,0}\big)_{\mathcal{W}}\big](q), \qquad \chi\big[\big(\mathcal{R}_{1,3}^{0,b}\big)_{\mathcal{W}}\big](q) = 1 + \chi\big[\big(\mathcal{R}_{1,6}^{0,3-b}\big)_{\mathcal{W}}\big](q) \tag{3.12}$$

and

$$\chi\big[\big(\mathcal{R}_{1,3\kappa}^{0,1}\big)_{\mathcal{W}}\big](q) + \chi\big[\big(\mathcal{R}_{1,3\kappa}^{0,2}\big)_{\mathcal{W}}\big](q) = \chi\big[\big(\mathcal{R}_{2\kappa,1}^{1,0}\big)_{\mathcal{W}}\big](q) + \chi\big[\big(\mathcal{R}_{2\kappa,2}^{1,0}\big)_{\mathcal{W}}\big](q). \tag{3.13}$$

The characters of the $\mathcal{W}$-indecomposable rank-3 representations read

$$\chi\big[\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}}\big](q) = 2 + \sum_{k\in\mathbb{N}}4k\mathrm{ch}_{2k+1,b}(q) + \sum_{k\in\mathbb{N}}8k\mathrm{ch}_{4k+1,3-b}(q) + \sum_{k\in\mathbb{N}}(8k-4)\mathrm{ch}_{4k-1,3-b}(q) \tag{3.14}$$

and are seen to be independent of κ. As we will discuss below, the dependence on κ manifests itself in the distinct Jordan-cell and general embedding structures of $\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}}$ for different $\kappa, b \in \mathbb{Z}_{1,2}$. Likewise, the $\mathcal{W}$-indecomposable rank-2 representations appearing in (3.11) have distinct embedding structures.

We also have $\mathcal{W}$-extended characters of the various *subfactors* of the $\mathcal{W}$-indecomposable representations

$$\begin{aligned}
\hat{\chi}_0(q) &= 1\\
\hat{\chi}_1(q) &= \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{4k-1,2}(q) = \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{1,6k-2}(q)\\
&= \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k^2\big(q^{(12k-7)^2/24}-q^{(12k+1)^2/24}\big)\\
\hat{\chi}_2(q) &= \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{4k-1,1}(q) = \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{1,6k-1}(q)\\
&= \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k^2\big(q^{(12k-5)^2/24}-q^{(12k-1)^2/24}\big)\\
\hat{\chi}_5(q) &= \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{4k+1,2}(q) = \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{1,6k+1}(q)\\
&= \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k(k+1)\big(q^{(12k-1)^2/24}-q^{(12k+7)^2/24}\big)\\
\hat{\chi}_7(q) &= \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{4k+1,1}(q) = \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{1,6k+2}(q)\\
&= \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k(k+1)\big(q^{(12k+1)^2/24}-q^{(12k+5)^2/24}\big).
\end{aligned}\tag{3.15}$$

Here we have used the notation $\hat{\chi}_\Delta(q)$, where Δ is the conformal dimension of the corresponding representation, and some of the identities

$$\begin{aligned}
&\Delta_{1,6k+2}=\Delta_{4k+1,1}, \quad \Delta_{1,6k+1}=\Delta_{4k+1,2}, \quad \Delta_{1,6k-1}=\Delta_{4k-1,1}, \quad \Delta_{1,6k-2}=\Delta_{4k-1,2}\\
&\Delta_{2,6k+2}=\Delta_{4k,1}, \quad \Delta_{2,6k+1}=\Delta_{4k,2}, \quad \Delta_{2,6k-1}=\Delta_{4k-2,1}, \quad \Delta_{2,6k-2}=\Delta_{4k-2,2}\\
&\Delta_{1,3k}=\Delta_{2k+1,3}, \quad \Delta_{2,3k}=\Delta_{2k,3}.
\end{aligned}\tag{3.16}$$

Similarly, written as $\hat{\chi}_\Delta(q)$, the eight independent characters in (3.9) read

$$\begin{aligned}
\hat{\chi}_{\frac{1}{3}}(q) &= \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{4k-1,3}(q) = \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{1,6k-3}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)q^{3(4k-3)^2/8}\\
\hat{\chi}_{\frac{10}{3}}(q) &= \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{4k+1,3}(q) = \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{1,6k}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2kq^{3(4k-1)^2/8}
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}
\hat{\chi}_{\frac{1}{8}}(q) &= \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{4k-2,2}(q) = \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{2,6k-2}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)q^{(6k-5)^2/6}\\
\hat{\chi}_{\frac{5}{8}}(q) &= \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{4k-2,1}(q) = \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{2,6k-1}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)q^{(6k-4)^2/6}\\
\hat{\chi}_{\frac{21}{8}}(q) &= \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{4k,2}(q) = \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{2,6k+1}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2kq^{(6k-2)^2/6}\\
\hat{\chi}_{\frac{33}{8}}(q) &= \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{4k,1}(q) = \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{2,6k+2}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2kq^{(6k-1)^2/6}
\end{aligned}\tag{3.18}$$

and

$$\hat{\chi}_{-\frac{1}{24}}(q) = \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{4k-2,3}(q) = \sum_{k\in\mathbb{N}}(2k-1)\mathrm{ch}_{2,6k-3}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)q^{(6k-6)^2/6}$$
$$\hat{\chi}_{\frac{35}{24}}(q) = \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{4k,3}(q) = \sum_{k\in\mathbb{N}}2k\mathrm{ch}_{2,6k}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2kq^{(6k-3)^2/6}. \tag{3.19}$$

We believe that the five characters in (3.15) and the eight characters in (3.17) through (3.19) correspond to $\mathcal{W}$-*irreducible* representations. This yields a total of 13 $\mathcal{W}$-irreducible representations. In terms of these irreducible characters, we have the decompositions

$$\begin{aligned}
\chi\big[\big(\mathcal{R}_{2,3}^{1,0}\big)_{\mathcal{W}}\big](q) &= \chi\big[\big(\mathcal{R}_{4,3}^{1,0}\big)_{\mathcal{W}}\big](q) = 2\hat{\chi}_{\frac{1}{3}}(q) + 2\hat{\chi}_{\frac{10}{3}}(q)\\
\chi\big[\big(\mathcal{R}_{2,3}^{0,1}\big)_{\mathcal{W}}\big](q) &= \chi\big[\big(\mathcal{R}_{2,6}^{0,2}\big)_{\mathcal{W}}\big](q) = 2\hat{\chi}_{\frac{1}{8}}(q) + 2\hat{\chi}_{\frac{33}{8}}(q)\\
\chi\big[\big(\mathcal{R}_{2,3}^{0,2}\big)_{\mathcal{W}}\big](q) &= \chi\big[\big(\mathcal{R}_{2,6}^{0,1}\big)_{\mathcal{W}}\big](q) = 2\hat{\chi}_{\frac{5}{8}}(q) + 2\hat{\chi}_{\frac{21}{8}}(q)
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\chi\big[\big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}}\big](q) &= 1 + \chi\big[\big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}}\big](q) = 1 + 2\hat{\chi}_2(q) + 2\hat{\chi}_7(q)\\
\chi\big[\big(\mathcal{R}_{2,2}^{1,0}\big)_{\mathcal{W}}\big](q) &= 1 + \chi\big[\big(\mathcal{R}_{4,2}^{1,0}\big)_{\mathcal{W}}\big](q) = 1 + 2\hat{\chi}_1(q) + 2\hat{\chi}_5(q)\\
\chi\big[\big(\mathcal{R}_{1,3}^{0,1}\big)_{\mathcal{W}}\big](q) &= 1 + \chi\big[\big(\mathcal{R}_{1,6}^{0,2}\big)_{\mathcal{W}}\big](q) = 1 + 2\hat{\chi}_1(q) + 2\hat{\chi}_7(q)\\
\chi\big[\big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}}\big](q) &= 1 + \chi\big[\big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}}\big](q) = 1 + 2\hat{\chi}_2(q) + 2\hat{\chi}_5(q)
\end{aligned} \tag{3.21}$$

and

$$\chi\big[\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}}\big](q) = 2 + 4\hat{\chi}_1(q) + 4\hat{\chi}_2(q) + 4\hat{\chi}_5(q) + 4\hat{\chi}_7(q). \tag{3.22}$$

The $\mathcal{W}$-irreducible representations whose characters are given by (3.15) are denoted below by $(\Delta)_{\mathcal{W}}$. Sometimes, we extend this practice to the $\mathcal{W}$-irreducible representations (3.9) as well. We refer to the finite Kac tables in figure 2 and 3 for natural organizations of the conformal weights of the 13 $\mathcal{W}$-irreducible representations.

Letting $\chi_{r,s}(q)$ denote the character of the Kac representation (r,s) where $r,s\in\mathbb{N}$, we have

$$\begin{aligned}
\hat{\chi}_0(q) &= \chi_{1,1}(q) - \sum_{k\in\mathbb{N}}(\chi_{4k-1,1}(q) - \chi_{4k+1,1}(q)) = \chi_{1,1}(q) - \sum_{k\in\mathbb{N}}(\chi_{1,6k-1}(q) - \chi_{1,6k+1}(q))\\
\hat{\chi}_1(q) &= \sum_{k\in\mathbb{N}}k^2(\chi_{4k-1,2}(q) - \chi_{4k+1,2}(q)) = \sum_{k\in\mathbb{N}}k^2(\chi_{1,6k-2}(q) - \chi_{1,6k+2}(q))\\
\hat{\chi}_2(q) &= \sum_{k\in\mathbb{N}}k^2(\chi_{4k-1,1}(q) - \chi_{4k+1,1}(q)) = \sum_{k\in\mathbb{N}}k^2(\chi_{1,6k-1}(q) - \chi_{1,6k+1}(q))\\
\hat{\chi}_5(q) &= \sum_{k\in\mathbb{N}}k(k+1)(\chi_{4k+1,2}(q) - \chi_{4(k+1)-1,2}(q)) = \sum_{k\in\mathbb{N}}k(k+1)(\chi_{1,6k+1}(q) - \chi_{1,6(k+1)-1}(q))\\
\hat{\chi}_7(q) &= \sum_{k\in\mathbb{N}}k(k+1)(\chi_{4k+1,1}(q) - \chi_{4(k+1)-1,1}(q)) = \sum_{k\in\mathbb{N}}k(k+1)(\chi_{1,6k+2}(q) - \chi_{1,6(k+1)-2}(q)).
\end{aligned} \tag{3.23}$$

Since the Kac representations appearing in (3.9) and (3.17) through (3.19) are *irreducible* Virasoro representations themselves, we have

$$\chi_{2(2k-2+\kappa),s}(q) = \mathrm{ch}_{2(2k-2+\kappa),s}(q), \qquad \chi_{r,3(2k-2+\kappa)}(q) = \mathrm{ch}_{r,3(2k-2+\kappa)}(q) \tag{3.24}$$

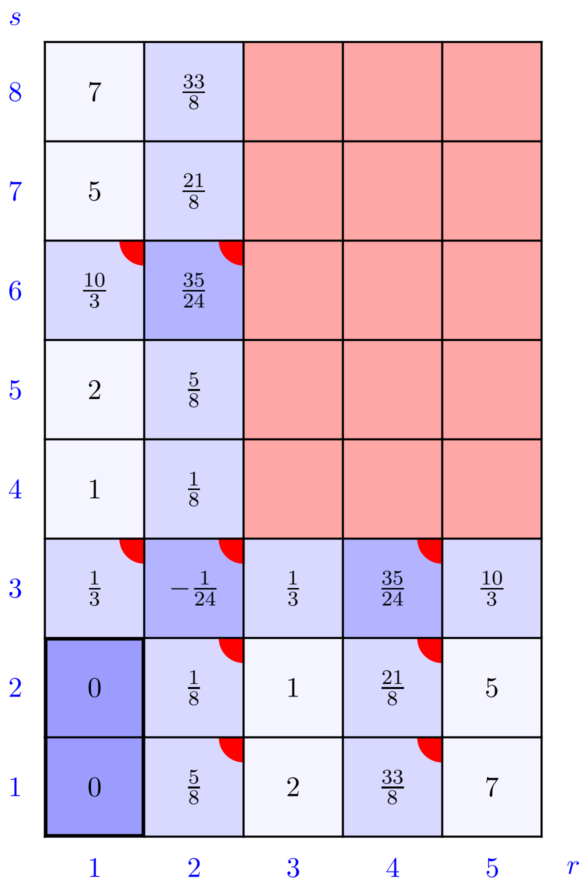


Figure 2. Finite part of the infinite Kac table of critical percolation. This part, which is relevant in the extended picture $\mathcal{WLM}(2, 3)$, corresponds to the bottom-left corner of the infinite Kac table of figure 1.

and hence

$$\hat{\chi}_{2\kappa,s}(q) = \sum_{k\in\mathbb{N}}(2k-2+\kappa)\chi_{2(2k-2+\kappa),s}(q),$$
$$\hat{\chi}_{r,3\kappa}(q) = \sum_{k\in\mathbb{N}}(2k-2+\kappa)\chi_{r,3(2k-2+\kappa)}(q). \tag{3.25}$$

3.2.1. Theta forms. The characters of the 13 $\mathcal{W}$-irreducible representations agree with those of [46]. In particular, they admit the expressions given there in terms of theta functions

$$\theta_{\ell,k}(q,z) = \sum_{j\in\mathbb{Z}+\frac{\ell}{2k}} q^{kj^2} z^{kj}, \qquad |q|<1, \qquad z\in\mathbb{C}, \qquad k\in\mathbb{N}, \qquad \ell\in\mathbb{Z} \tag{3.26}$$

and theta constants

$$\theta_{\ell,k}(q) = \theta_{\ell,k}(q,1), \qquad \theta_{\ell,k}^{(m)}(q) = \left(z\frac{\partial}{\partial z}\right)^m \theta_{\ell,k}(q,z)\bigg|_{z=1}, \qquad m\in\mathbb{N}. \tag{3.27}$$

Introducing the abbreviations

$$\theta_\ell(q) = \theta_{\ell,pp'}(q), \qquad \theta'_\ell(q) = \theta^{(1)}_{\ell,pp'}(q), \qquad \theta''_\ell(q) = \theta^{(2)}_{\ell,pp'}(q), \tag{3.28}$$

the theta forms are

$$\hat{\chi}_{r,s}(q) = \frac{1}{\eta(q)}(\theta_{sp-rp'}(q) - \theta_{sp+rp'}(q)), \qquad r \in \mathbb{Z}_{1,p-1}, \qquad s \in \mathbb{Z}_{1,p'-1}, \qquad sp+rp' \leqslant pp' \tag{3.29}$$

$$\hat{\chi}^+_{r,s}(q) = \frac{1}{(pp')^2\eta(q)}\bigg(\theta''_{sp+rp'}(q) - \theta''_{sp-rp'}(q) - (sp+rp')\theta'_{sp+rp'}(q) + (sp-rp')\theta'_{sp-rp'}(q) + \frac{(sp+rp')^2}{4}\theta_{sp+rp'}(q) - \frac{(sp-rp')^2}{4}\theta_{sp-rp'}(q)\bigg), \qquad r \in \mathbb{Z}_{1,p}, \quad s \in \mathbb{Z}_{1,p'} \tag{3.30}$$

$$\hat{\chi}^-_{r,s}(q) = \frac{1}{(pp')^2\eta(q)}\bigg(\theta''_{pp'-sp-rp'}(q) - \theta''_{pp'+sp-rp'}(q) + (sp+rp')\theta'_{pp'-sp-rp'}(q) + (sp-rp')\theta'_{pp'+sp-rp'}(q) + \frac{(sp+rp')^2-(pp')^2}{4}\theta_{pp'-sp-rp'}(q) - \frac{(sp-rp')^2-(pp')^2}{4}\theta_{pp'+sp-rp'}(q)\bigg), \qquad r \in \mathbb{Z}_{1,p}, \qquad s \in \mathbb{Z}_{1,p'}, \tag{3.31}$$

where the Dedekind eta function is defined in (2.4). As the notation suggests, these are believed to be the theta forms relevant in the case of general p, p' [46]. It is noted that the theta form $\hat{\chi}_{r,s}(q)$ is identical to the well-known irreducible Virasoro minimal character $\chi_{\Delta_{r,s}}(q) = \text{ch}_{r,s}(q)$. The precise relations between our $\mathcal{W}$-irreducible characters and the theta forms for $p = 2$ and $p' = 3$ are

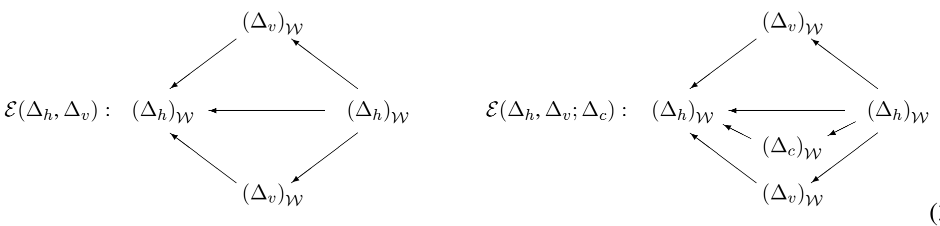
$$\hat{\chi}_0(q) = \hat{\chi}_{1,1}(q) = 1, \qquad \begin{array}{l}\hat{\chi}_1(q) = \hat{\chi}^+_{1,2}(q), \\ \hat{\chi}_2(q) = \hat{\chi}^+_{1,1}(q),\end{array} \qquad \begin{array}{l}\hat{\chi}_5(q) = \hat{\chi}^-_{1,2}(q) \\ \hat{\chi}_7(q) = \hat{\chi}^-_{1,1}(q)\end{array} \tag{3.32}$$

$$\hat{\chi}_{2\kappa,s}(q) = \begin{cases}\hat{\chi}^+_{2,s}(q), & \kappa = 1 \\ \hat{\chi}^-_{2,s}(q), & \kappa = 2\end{cases} \qquad \hat{\chi}_{r,3\kappa}(q) = \begin{cases}\hat{\chi}^+_{r,3}(q), & \kappa = 1 \\ \hat{\chi}^-_{r,3}(q), & \kappa = 2.\end{cases} \tag{3.33}$$

The $\mathcal{W}$-irreducible characters $\hat{\chi}_{2,3}(q)$ and $\hat{\chi}_{4,3}(q) = \hat{\chi}_{2,6}(q)$ are listed twice. A compact version of the Kac table in figure 2 is given in figure 3.

3.3. Embedding diagrams and Jordan-cell structures

We conjecture that every $\mathcal{W}$-indecomposable rank-2 representation has an embedding pattern of one of the types



(3.34)

where the horizontal arrows indicate the off-diagonal action of the Virasoro mode L_0. Specifically, we conjecture that the 14 $\mathcal{W}$-indecomposable rank-2 representations (3.2) enjoy the embedding patterns

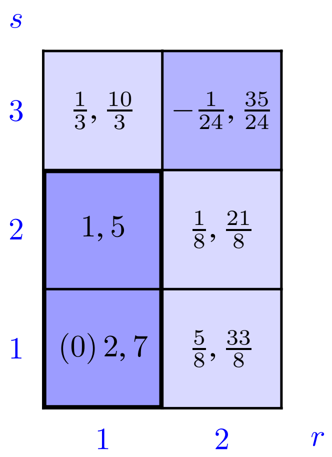


Figure 3. Schematic finite Kac table, following [46], of the 13 $\mathcal{W}$-irreducible representations for critical percolation in the extended picture $\mathcal{WLM}(2,3)$.

$$
\begin{aligned}
&\big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \sim \mathcal{E}(2,7;0), && \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}} \sim \mathcal{E}(7,2), && \big(\mathcal{R}_{2,3}^{1,0}\big)_{\mathcal{W}} \sim \mathcal{E}\big(\tfrac{1}{3},\tfrac{10}{3}\big), && \big(\mathcal{R}_{4,3}^{1,0}\big)_{\mathcal{W}} \sim \mathcal{E}\big(\tfrac{10}{3},\tfrac{1}{3}\big)\\
&\big(\mathcal{R}_{2,2}^{1,0}\big)_{\mathcal{W}} \sim \mathcal{E}(1,5;0), && \big(\mathcal{R}_{4,2}^{1,0}\big)_{\mathcal{W}} \sim \mathcal{E}(5,1), && \big(\mathcal{R}_{2,3}^{0,1}\big)_{\mathcal{W}} \sim \mathcal{E}\big(\tfrac{1}{8},\tfrac{33}{8}\big), && \big(\mathcal{R}_{2,6}^{0,2}\big)_{\mathcal{W}} \sim \mathcal{E}\big(\tfrac{33}{8},\tfrac{1}{8}\big)\\
&\big(\mathcal{R}_{1,3}^{0,1}\big)_{\mathcal{W}} \sim \mathcal{E}(1,7;0), && \big(\mathcal{R}_{1,6}^{0,2}\big)_{\mathcal{W}} \sim \mathcal{E}(7,1), && \big(\mathcal{R}_{2,3}^{0,2}\big)_{\mathcal{W}} \sim \mathcal{E}\big(\tfrac{5}{8},\tfrac{21}{8}\big), && \big(\mathcal{R}_{2,6}^{0,1}\big)_{\mathcal{W}} \sim \mathcal{E}\big(\tfrac{21}{8},\tfrac{5}{8}\big)\\
&\big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}} \sim \mathcal{E}(2,5;0), && \big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}} \sim \mathcal{E}(5,2).
\end{aligned}
\tag{3.35}
$$

We can encode the Jordan-cell structure of a $\mathcal{W}$-indecomposable rank-2 representation in its character by introducing the matrix

$$
\mathcal{J}_2 = \begin{pmatrix} 1 & 1\\ 0 & 1 \end{pmatrix}. \tag{3.36}
$$

Its trace is simply $\mathrm{Tr}(\mathcal{J}_2) = 2$ but can be used to indicate the presence of Jordan cells of rank 2. By

$$
\mathrm{Tr}(\mathcal{J}_2)(\mathrm{ch}_{r,s}(q) + \mathrm{ch}_{r',s'}(q)) + 2\mathrm{ch}_{r'',s''}(q), \tag{3.37}
$$

we thus mean a sum of six irreducible characters where a Jordan cell of rank 2 is formed between every pair of matching states in the two modules labelled by r, s and between every pair of matching states in the two modules labelled by r', s' while no state in the modules labelled by r'', s'' is part of a non-trivial Jordan cell. The characters of the $\mathcal{W}$-indecomposable rank-2 representations then read

$$
\begin{aligned}
\chi\big[\big(\mathcal{R}_{2\kappa,s}^{1,0}\big)_{\mathcal{W}}\big](q) &= \delta_{\kappa,1}\{1-\delta_{s,3}\} + 2\sum_{k\in\mathbb{N}}(2k+1-\kappa)\mathrm{ch}_{4k+3-2\kappa,s}(q)\\
&\quad + \mathrm{Tr}(\mathcal{J}_2)\sum_{k\in\mathbb{N}}(2k-2+\kappa)\mathrm{ch}_{4k-3+2\kappa,s}(q)\\
\chi\big[\big(\mathcal{R}_{r,3\kappa}^{0,b}\big)_{\mathcal{W}}\big](q) &= \delta_{\kappa,1}\{1-\delta_{r,2}\} + 2\sum_{k\in\mathbb{N}}(2k+1-\kappa)\mathrm{ch}_{r,6k+6-3\kappa-b}(q)\\
&\quad + \mathrm{Tr}(\mathcal{J}_2)\sum_{k\in\mathbb{N}}(2k-2+\kappa)\mathrm{ch}_{r,6k-6+3\kappa+b}(q).
\end{aligned}
\tag{3.38}
$$

These refined character expressions demonstrate the inequivalence of the various representations despite the character identities (3.11). The relations (3.13) are valid for the refined characters as well, whereas the relations (3.12) are not. We note that the refined character expressions contain enough information to distinguish between the different rank-2 representations. That is, the distinctions can be made by solely emphasizing the Jordan-cell structures without further reference to the complete embedding patterns.

Similar refinements of the rank-3 characters are possible (see below) but not required to demonstrate inequivalence of the associated $\mathcal{W}$-indecomposable rank-3 representations. Indeed, it suffices to focus on the presence of rank-3 Jordan cells to which end we introduce the matrix

$$\mathcal{J}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.39}$$

with trace $\mathrm{Tr}(\mathcal{J}_3) = 3$. Ignoring Jordan cells of rank 2 all together, the 'semi-refined' characters of the $\mathcal{W}$-indecomposable rank-3 representations then read

$$\begin{aligned}\chi\big[\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}}\big](q) &= 2 + 4\sum_{k\in\mathbb{N}} k\,\mathrm{ch}_{2k+1,b}(q) + 4\sum_{k\in\mathbb{N}}(2k+1-\kappa)\mathrm{ch}_{4k+3-2\kappa,3-b}(q) \\ &\quad + \{\mathrm{Tr}(\mathcal{J}_3)+1\}\sum_{k\in\mathbb{N}}(2k-2+\kappa)\mathrm{ch}_{4k-3+2\kappa,3-b}(q).\end{aligned} \tag{3.40}$$

With $\kappa, b \in \mathbb{Z}_{1,2}$, these four semi-refined characters correspond to four distinct representations despite the character identities implicit in (3.14).

We conclude this discussion of embedding patterns by conjecturing that the $\mathcal{W}$-indecomposable rank-3 representations also have embedding structures described by the patterns in (3.34). Specifically, we conjecture that

$$\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}} \sim \mathcal{E}\big(\big(\mathcal{R}_{2\kappa,3-b}^{1,0}\big)_{\mathcal{W}}, \big(\mathcal{R}_{2(3-\kappa),b}^{1,0}\big)_{\mathcal{W}}\big) \sim \mathcal{E}\big(\big(\mathcal{R}_{1,3\kappa}^{0,b}\big)_{\mathcal{W}}, \big(\mathcal{R}_{1,3(3-\kappa)}^{0,b}\big)_{\mathcal{W}}\big), \tag{3.41}$$

where the $\mathcal{W}$-irreducible representations $(\Delta_h)_{\mathcal{W}}$ and $(\Delta_v)_{\mathcal{W}}$ have been replaced by $\mathcal{W}$-indecomposable rank-2 representations. It is noted that each of the four rank-3 representations is thus proposed to be viewable in two different ways. This corresponds to viewing it as an indecomposable 'vertical' combination of 'horizontal' rank-2 representations $(\mathcal{R}^{1,0})_{\mathcal{W}}$ or as an indecomposable 'horizontal' combination of 'vertical' rank-2 representations $(\mathcal{R}^{0,b})_{\mathcal{W}}$. Converting the two rank-2 Jordan cells linked by a horizontal arrow into a rank-3 and a rank-1 Jordan cell, we finally arrive at the announced refined characters

$$\begin{aligned}\chi\big[\big(\mathcal{R}_{2,3}^{1,1}\big)_{\mathcal{W}}\big](q) &= \mathrm{Tr}(\mathcal{J}_2)\hat\chi_0(q) + \{\mathrm{Tr}(\mathcal{J}_3)+1\}\hat\chi_1(q) + 4\hat\chi_2(q) + 2\mathrm{Tr}(\mathcal{J}_2)\hat\chi_5(q) + 2\mathrm{Tr}(\mathcal{J}_2)\hat\chi_7(q) \\ \chi\big[\big(\mathcal{R}_{4,3}^{1,1}\big)_{\mathcal{W}}\big](q) &= 2\hat\chi_0(q) + 2\mathrm{Tr}(\mathcal{J}_2)\hat\chi_1(q) + 2\mathrm{Tr}(\mathcal{J}_2)\hat\chi_2(q) + \{\mathrm{Tr}(\mathcal{J}_3)+1\}\hat\chi_5(q) + 4\hat\chi_7(q) \\ \chi\big[\big(\mathcal{R}_{2,3}^{1,2}\big)_{\mathcal{W}}\big](q) &= \mathrm{Tr}(\mathcal{J}_2)\hat\chi_0(q) + 4\hat\chi_1(q) + \{\mathrm{Tr}(\mathcal{J}_3)+1\}\hat\chi_2(q) + 2\mathrm{Tr}(\mathcal{J}_2)\hat\chi_5(q) + 2\mathrm{Tr}(\mathcal{J}_2)\hat\chi_7(q) \\ \chi\big[\big(\mathcal{R}_{4,3}^{1,2}\big)_{\mathcal{W}}\big](q) &= 2\hat\chi_0(q) + 2\mathrm{Tr}(\mathcal{J}_2)\hat\chi_1(q) + 2\mathrm{Tr}(\mathcal{J}_2)\hat\chi_2(q) + 4\hat\chi_5(q) + \{\mathrm{Tr}(\mathcal{J}_3)+1\}\hat\chi_7(q),\end{aligned} \tag{3.42}$$

here expressed explicitly in terms of the $\mathcal{W}$-irreducible characters (3.15).

3.4. Summary of $\mathcal{W}$-extended fusion algebra

We denote the fusion product in the $\mathcal{W}$-extended picture by $\hat\otimes$ and reserve the symbol $\otimes$ for the fusion product in the Virasoro picture. A summary of the fusion algebra of critical percolation in the $\mathcal{W}$-extended picture $\mathcal{WLM}(2,3)$ is given in the following. It is both associative and

$\hat{\otimes}$	rank 1	rank 2	rank 3
rank 1	F_5	U_6^T	L_6^T
rank 2	U_6	$(U_7\vert U_8)$	$(L_7\vert L_8)^T$
rank 3	L_6	$(L_7\vert L_8)$	F_9

Figure 4. Schematic Cayley table of the $\mathcal{W}$-extended fusion algebra of critical percolation. Here F_j corresponds to the table given in figure j while F_j^T corresponds to the transpose thereof. By U_j (L_j) we mean the 'upper' ('lower') part of the table in figure j while $(U_7|U_8)$ is the horizontal concatenation of the tables U_7 and U_8.

commutative. To compactify the results a bit, we introduce the following linear combinations:

$$\begin{aligned} \mathcal{C}_s &= 2(2,s)_{\mathcal{W}} \oplus 2(4,s)_{\mathcal{W}}, & \mathcal{C}_s^{1,0} &= 2\big(\mathcal{R}_{2,s}^{1,0}\big)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{4,s}^{1,0}\big)_{\mathcal{W}} \\ \mathcal{C}^{0,b} &= 2\big(\mathcal{R}_{2,3}^{0,b}\big)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{2,6}^{0,b}\big)_{\mathcal{W}}, & \mathcal{C}^{1,b} &= 2\big(\mathcal{R}_{2,3}^{1,b}\big)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{4,3}^{1,b}\big)_{\mathcal{W}} \\ \hat{\mathcal{C}}^0 &= 4\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1} \oplus 2\mathcal{C}^{0,2}, & \hat{\mathcal{C}}^1 &= 4\mathcal{C}_3^{1,0} \oplus 2\mathcal{C}^{1,1} \oplus 2\mathcal{C}^{1,2} \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} \mathcal{D}_{\kappa,3\kappa'}^{0,b} &= 2(\kappa, 3(3-b\cdot\kappa'))_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{\kappa,3\kappa'}^{0,b}\big)_{\mathcal{W}}, \\ \mathcal{D}_{2\kappa,3}^{1,b} &= 2\big(\mathcal{R}_{2(3-b\cdot\kappa),3}^{1,0}\big)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}}, \end{aligned} \tag{3.44}$$

where it is recalled that $(2,6)_{\mathcal{W}} \equiv (4,3)_{\mathcal{W}}$ and where

$$m\cdot n = \frac{3-(-1)^{m+n}}{2}, \qquad m,n\in\mathbb{Z}. \tag{3.45}$$

The fusion rules are listed in the tables in figures 5–9. They are easily combined to form a complete Cayley table as indicated in figure 4. All entries of the Cayley table of the fusions of $\mathcal{W}$-indecomposable rank-3 representations provided in figure 9 are given by

$$\hat{\mathcal{C}}^1 = 8\big(\mathcal{R}_{2,3}^{1,0}\big)_{\mathcal{W}} \oplus 8\big(\mathcal{R}_{4,3}^{1,0}\big)_{\mathcal{W}} \oplus 4\big(\mathcal{R}_{2,3}^{1,1}\big)_{\mathcal{W}} \oplus 4\big(\mathcal{R}_{4,3}^{1,1}\big)_{\mathcal{W}} \oplus 4\big(\mathcal{R}_{2,3}^{1,2}\big)_{\mathcal{W}} \oplus 4\big(\mathcal{R}_{4,3}^{1,2}\big)_{\mathcal{W}}. \tag{3.46}$$

It is noted that the fusion algebra just listed does not contain an identity. We will discuss this further in section 5.

4. Lattice realization of $\mathcal{WLM}(2,3)$

In [44], we used the infinite series of logarithmic minimal lattice models $\mathcal{LM}(1,p)$ to obtain $\mathcal{W}$-extended fusion rules applicable in the extended pictures $\mathcal{WLM}(1,p)$. A crucial ingredient was the construction of a $\mathcal{W}$-invariant identity representation $(1,1)_{\mathcal{W}}$ defined as the infinite limit of a triple fusion of Virasoro-irreducible Kac representations in $\mathcal{LM}(1,p)$. On the other hand, as indicated above and further discussed in section 5, there is no obvious natural candidate for an identity in the lattice realization of $\mathcal{WLM}(2,3)$. It nevertheless turns out fruitful to adopt the use of infinite limits of triple fusions of Virasoro-irreducible Kac representations. This also allows us to identify the various $\mathcal{W}$-representations with suitable limits of Yang–Baxter integrable boundary conditions on the lattice. Firmly based on the lattice realization of the fundamental fusion algebra of $\mathcal{LM}(2,3)$, our fusion prescription for $\mathcal{WLM}(2,3)$ yields a commutative and associative fusion algebra.

$\hat{\otimes}$	$(2,1)_{\mathcal{W}}$	$(4,1)_{\mathcal{W}}$	$(2,2)_{\mathcal{W}}$	$(4,2)_{\mathcal{W}}$	$(1,3)_{\mathcal{W}}$	$(1,6)_{\mathcal{W}}$	$(2,3)_{\mathcal{W}}$	$(4,3)_{\mathcal{W}}$
$(2,1)_{\mathcal{W}}$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,2}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,2}^{1,0})_{\mathcal{W}}$	$(2,3)_{\mathcal{W}}$	$(4,3)_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$
$(4,1)_{\mathcal{W}}$	$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,2}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,2}^{1,0})_{\mathcal{W}}$	$(4,3)_{\mathcal{W}}$	$(2,3)_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$
$(2,2)_{\mathcal{W}}$	$(\mathcal{R}_{2,2}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,2}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$
$(4,2)_{\mathcal{W}}$	$(\mathcal{R}_{4,2}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,2}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$
$(1,3)_{\mathcal{W}}$	$(2,3)_{\mathcal{W}}$	$(4,3)_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$(1,3)_{\mathcal{W}} \oplus (\mathcal{R}_{1,3}^{0,2})_{\mathcal{W}}$	$(1,6)_{\mathcal{W}} \oplus (\mathcal{R}_{1,6}^{0,2})_{\mathcal{W}}$	$(2,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$(4,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$
$(1,6)_{\mathcal{W}}$	$(4,3)_{\mathcal{W}}$	$(2,3)_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$(1,6)_{\mathcal{W}} \oplus (\mathcal{R}_{1,6}^{0,2})_{\mathcal{W}}$	$(1,3)_{\mathcal{W}} \oplus (\mathcal{R}_{1,3}^{0,2})_{\mathcal{W}}$	$(4,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$(2,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$
$(2,3)_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(2,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$(4,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$
$(4,3)_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(4,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$(2,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$

Figure 5. Cayley table of the fusions of $\mathcal{W}$-indecomposable rank-1 representations with $\mathcal{W}$-indecomposable rank-1 representations.

$\hat{\otimes}$	$(2,1)_{\mathcal{W}}$	$(4,1)_{\mathcal{W}}$	$(2,2)_{\mathcal{W}}$	$(4,2)_{\mathcal{W}}$	$(1,3)_{\mathcal{W}}$	$(1,6)_{\mathcal{W}}$	$(2,3)_{\mathcal{W}}$	$(4,3)_{\mathcal{W}}$
$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_2$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_3$	$\mathcal{C}_3$
$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_2$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_3$	$\mathcal{C}_3$
$(\mathcal{R}_{2,2}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_2$	$\mathcal{C}_2$	$\mathcal{C}_1 \oplus \mathcal{C}_3$	$\mathcal{C}_1 \oplus \mathcal{C}_3$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$
$(\mathcal{R}_{4,2}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_2$	$\mathcal{C}_2$	$\mathcal{C}_1 \oplus \mathcal{C}_3$	$\mathcal{C}_1 \oplus \mathcal{C}_3$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$
$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_3$	$\mathcal{C}_3$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$
$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_3$	$\mathcal{C}_3$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$
$(\mathcal{R}_{1,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$2(2,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$2(4,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$\mathcal{D}_{1,3}^{0,1}$	$\mathcal{D}_{1,6}^{0,1}$	$\mathcal{D}_{2,3}^{0,1}$	$\mathcal{D}_{2,6}^{0,1}$
$(\mathcal{R}_{1,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$2(4,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$2(2,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$\mathcal{D}_{1,6}^{0,1}$	$\mathcal{D}_{1,3}^{0,1}$	$\mathcal{D}_{2,6}^{0,1}$	$\mathcal{D}_{2,3}^{0,1}$
$(\mathcal{R}_{1,3}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$2(4,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$2(2,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$\mathcal{D}_{1,6}^{0,1}$	$\mathcal{D}_{1,3}^{0,1}$	$\mathcal{D}_{2,6}^{0,1}$	$\mathcal{D}_{2,3}^{0,1}$
$(\mathcal{R}_{1,6}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$2(2,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$2(4,3)_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$\mathcal{D}_{1,3}^{0,1}$	$\mathcal{D}_{1,6}^{0,1}$	$\mathcal{D}_{2,3}^{0,1}$	$\mathcal{D}_{2,6}^{0,1}$
$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{0,1}$	$\mathcal{D}_{2,6}^{0,1}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$
$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$\mathcal{D}_{2,6}^{0,1}$	$\mathcal{D}_{2,3}^{0,1}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$
$(\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$\mathcal{D}_{2,6}^{0,1}$	$\mathcal{D}_{2,3}^{0,1}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$
$(\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{0,1}$	$\mathcal{D}_{2,6}^{0,1}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$
$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$
$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$
$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$\mathcal{C}^{0,2}$	$\mathcal{C}^{0,2}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,1}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$
$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$\mathcal{C}^{0,2}$	$\mathcal{C}^{0,2}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,1}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$

Figure 6. Table of the fusions of $\mathcal{W}$-indecomposable rank-1 representations with $\mathcal{W}$-indecomposable rank-2 or rank-3 representations.

$\hat{\otimes}$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,2}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,2}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$
$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_1^{1,0}$	$\mathcal{C}_1^{1,0}$	$\mathcal{C}_2^{1,0}$	$\mathcal{C}_2^{1,0}$	$\mathcal{C}_3^{1,0}$	$\mathcal{C}_3^{1,0}$
$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_1^{1,0}$	$\mathcal{C}_1^{1,0}$	$\mathcal{C}_2^{1,0}$	$\mathcal{C}_2^{1,0}$	$\mathcal{C}_3^{1,0}$	$\mathcal{C}_3^{1,0}$
$(\mathcal{R}_{2,2}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_2^{1,0}$	$\mathcal{C}_2^{1,0}$	$\mathcal{C}_1^{1,0}\oplus\mathcal{C}_3^{1,0}$	$\mathcal{C}_1^{1,0}\oplus\mathcal{C}_3^{1,0}$	$\mathcal{C}^{1,1}$	$\mathcal{C}^{1,1}$
$(\mathcal{R}_{4,2}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_2^{1,0}$	$\mathcal{C}_2^{1,0}$	$\mathcal{C}_1^{1,0}\oplus\mathcal{C}_3^{1,0}$	$\mathcal{C}_1^{1,0}\oplus\mathcal{C}_3^{1,0}$	$\mathcal{C}^{1,1}$	$\mathcal{C}^{1,1}$
$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_3^{1,0}$	$\mathcal{C}_3^{1,0}$	$\mathcal{C}^{1,1}$	$\mathcal{C}^{1,1}$	$\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,2}$	$\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,2}$
$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$\mathcal{C}_3^{1,0}$	$\mathcal{C}_3^{1,0}$	$\mathcal{C}^{1,1}$	$\mathcal{C}^{1,1}$	$\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,2}$	$\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,2}$
$(\mathcal{R}_{1,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}\oplus(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}\oplus(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$
$(\mathcal{R}_{1,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}\oplus(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}\oplus(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$
$(\mathcal{R}_{1,3}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}\oplus(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}\oplus(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$
$(\mathcal{R}_{1,6}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}\oplus(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}\oplus(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$
$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus\mathcal{C}^{0,2}$	$2\mathcal{C}_3\oplus\mathcal{C}^{0,2}$	$2\mathcal{C}_3\oplus2\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus2\mathcal{C}^{0,1}$
$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus\mathcal{C}^{0,2}$	$2\mathcal{C}_3\oplus\mathcal{C}^{0,2}$	$2\mathcal{C}_3\oplus2\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus2\mathcal{C}^{0,1}$
$(\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$\mathcal{C}^{0,2}$	$\mathcal{C}^{0,2}$	$2\mathcal{C}_3\oplus\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus2\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus2\mathcal{C}^{0,1}$
$(\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$\mathcal{C}^{0,2}$	$\mathcal{C}^{0,2}$	$2\mathcal{C}_3\oplus\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus2\mathcal{C}^{0,1}$	$2\mathcal{C}_3\oplus2\mathcal{C}^{0,1}$
$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$\mathcal{C}^{1,1}$	$\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,2}$	$2\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,2}$	$2\mathcal{C}_3^{1,0}\oplus2\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus2\mathcal{C}^{1,1}$
$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$\mathcal{C}^{1,1}$	$\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,2}$	$2\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,2}$	$2\mathcal{C}_3^{1,0}\oplus2\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus2\mathcal{C}^{1,1}$
$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$\mathcal{C}^{1,2}$	$\mathcal{C}^{1,2}$	$2\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus2\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus2\mathcal{C}^{1,1}$
$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$\mathcal{C}^{1,2}$	$\mathcal{C}^{1,2}$	$2\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus2\mathcal{C}^{1,1}$	$2\mathcal{C}_3^{1,0}\oplus2\mathcal{C}^{1,1}$

Figure 7. First part of the table of the fusions of $\mathcal{W}$-indecomposable rank-2 representations with $\mathcal{W}$-indecomposable rank-2 or rank-3 representations.

4.1. *Horizontal component*

Working in the *fundamental* fusion algebra of critical percolation $\mathcal{LM}(2,3)$, as opposed to the less understood but larger *full* fusion algebra [34, 35], the only horizontal Kac representations at our disposal are $\{(2k,1); k\in\mathbb{N}\}$. It is noted that these are all Virasoro-irreducible representations. There are many possible triple fusions to consider of which the following one offers fairly straightforward access to the $\mathcal{W}$-extended horizontal component:

$$\lim_{n\to\infty}(4n,1)^{\otimes 3}=\bigoplus_{k\in\mathbb{N}}2k(2k,1)=2\left(\bigoplus_{k\in\mathbb{N}}(2k-1)(4k-2,1)\right)\oplus 2\left(\bigoplus_{k\in\mathbb{N}}2k(4k,1)\right). \quad (4.1)$$

$\hat{\otimes}$	$(\mathcal{R}_{1,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{1,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{1,3}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{1,6}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$
$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,2}$	$\mathcal{C}^{0,2}$
$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,1}$	$\mathcal{C}^{0,2}$	$\mathcal{C}^{0,2}$
$(\mathcal{R}_{2,2}^{1,0})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,1}$
$(\mathcal{R}_{4,2}^{1,0})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$2(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$2(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}} \oplus (\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,2}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus \mathcal{C}^{0,1}$
$(\mathcal{R}_{2,3}^{1,0})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$
$(\mathcal{R}_{4,3}^{1,0})_{\mathcal{W}}$	$\mathcal{D}_{4,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{2,3}^{1,1}$	$\mathcal{D}_{4,3}^{1,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$	$2\mathcal{C}_3 \oplus 2\mathcal{C}^{0,1}$
$(\mathcal{R}_{1,3}^{0,1})_{\mathcal{W}}$	$\mathcal{D}_{1,6}^{0,1} \oplus \mathcal{D}_{1,3}^{0,2}$	$\mathcal{D}_{1,3}^{0,1} \oplus \mathcal{D}_{1,6}^{0,2}$	$\mathcal{D}_{1,3}^{0,1} \oplus \mathcal{D}_{1,6}^{0,2}$	$\mathcal{D}_{1,6}^{0,1} \oplus \mathcal{D}_{1,3}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$
$(\mathcal{R}_{1,6}^{0,1})_{\mathcal{W}}$	$\mathcal{D}_{1,3}^{0,1} \oplus \mathcal{D}_{1,6}^{0,2}$	$\mathcal{D}_{1,6}^{0,1} \oplus \mathcal{D}_{1,3}^{0,2}$	$\mathcal{D}_{1,6}^{0,1} \oplus \mathcal{D}_{1,3}^{0,2}$	$\mathcal{D}_{1,3}^{0,1} \oplus \mathcal{D}_{1,6}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$
$(\mathcal{R}_{1,3}^{0,2})_{\mathcal{W}}$	$\mathcal{D}_{1,3}^{0,1} \oplus \mathcal{D}_{1,6}^{0,2}$	$\mathcal{D}_{1,6}^{0,1} \oplus \mathcal{D}_{1,3}^{0,2}$	$\mathcal{D}_{1,6}^{0,1} \oplus \mathcal{D}_{1,3}^{0,2}$	$\mathcal{D}_{1,3}^{0,1} \oplus \mathcal{D}_{1,6}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$
$(\mathcal{R}_{1,6}^{0,2})_{\mathcal{W}}$	$\mathcal{D}_{1,6}^{0,1} \oplus \mathcal{D}_{1,3}^{0,2}$	$\mathcal{D}_{1,3}^{0,1} \oplus \mathcal{D}_{1,6}^{0,2}$	$\mathcal{D}_{1,3}^{0,1} \oplus \mathcal{D}_{1,6}^{0,2}$	$\mathcal{D}_{1,6}^{0,1} \oplus \mathcal{D}_{1,3}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$
$(\mathcal{R}_{2,3}^{0,1})_{\mathcal{W}}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$
$(\mathcal{R}_{2,6}^{0,1})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$
$(\mathcal{R}_{2,3}^{0,2})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$
$(\mathcal{R}_{2,6}^{0,2})_{\mathcal{W}}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,3}^{0,1} \oplus \mathcal{D}_{2,6}^{0,2}$	$\mathcal{D}_{2,6}^{0,1} \oplus \mathcal{D}_{2,3}^{0,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$
$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$
$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$
$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$
$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{2,3}^{1,1} \oplus \mathcal{D}_{4,3}^{1,2}$	$\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$	$\hat{\mathcal{C}}^0$

Figure 8. Second part of the table of the fusions of $\mathcal{W}$-indecomposable rank-2 representations with $\mathcal{W}$-indecomposable rank-2 or rank-3 representations.

$\hat{\otimes}$	$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$
$(\mathcal{R}_{2,3}^{1,1})_{\mathcal{W}}$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$
$(\mathcal{R}_{4,3}^{1,1})_{\mathcal{W}}$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$
$(\mathcal{R}_{2,3}^{1,2})_{\mathcal{W}}$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$
$(\mathcal{R}_{4,3}^{1,2})_{\mathcal{W}}$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$	$\hat{\mathcal{C}}^1$

Figure 9. Cayley table of the fusions of $\mathcal{W}$-indecomposable rank-3 representations with $\mathcal{W}$-indecomposable rank-3 representations.

Indeed, we now assert that this limit corresponds to the following direct sum of four $\mathcal{W}$-indecomposable representations:

$$2(2,1)_{\mathcal{W}} \oplus 2(4,1)_{\mathcal{W}} := \lim_{n\to\infty} (4n,1)^{\otimes 3}, \tag{4.2}$$

whose decompositions in terms of Virasoro-irreducible representations read

$$(2,1)_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} (2k-1)(4k-2,1), \qquad (4,1)_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} 2k(4k,1). \tag{4.3}$$

Since the participating Virasoro representations all are of rank 1, the $\mathcal{W}$-indecomposable representations $(2,1)_{\mathcal{W}}$ and $(4,1)_{\mathcal{W}}$ themselves are of rank 1.

Without going into details, this separation or disentanglement of the triple fusion into four $\mathcal{W}$-indecomposable representations can be made manifest from the lattice by separating the set of link states accordingly. Since no non-trivial Jordan cells are formed between the representations on the right-hand side of (4.1), selecting the link states associated with either $(2,1)_{\mathcal{W}}$ or $(4,1)_{\mathcal{W}}$ is a valid procedure. When non-trivial Jordan cells are involved, on the other hand, such a selection may affect the distribution and ranks of the cells and hence would not be valid.

Having identified $(2,1)_{\mathcal{W}}$ and $(4,1)_{\mathcal{W}}$, we now define the $\mathcal{W}$-indecomposable rank-2 representations

$$\big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} := (2,1) \otimes (2,1)_{\mathcal{W}}, \qquad \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}} := (2,1) \otimes (4,1)_{\mathcal{W}}. \tag{4.4}$$

Their decompositions into Virasoro-indecomposable rank-2 representations are given in (3.6). Of importance for the evaluation of fusion products below, we note that the $\mathcal{W}$-indecomposable representations (4.3) and (4.4) have the stability properties

$$(4n,1) \otimes (2\kappa,1)_{\mathcal{W}} = 2n\big(\mathcal{R}_{2(3-\kappa),1}^{1,0}\big)_{\mathcal{W}}, \qquad (4n,1) \otimes \big(\mathcal{R}_{2\kappa,1}^{1,0}\big)_{\mathcal{W}} = 4n(2,1)_{\mathcal{W}} \oplus 4n(4,1)_{\mathcal{W}} \tag{4.5}$$

and

$$(2,1) \otimes \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} = (2,1) \otimes \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}} = 2(2,1)_{\mathcal{W}} \oplus 2(4,1)_{\mathcal{W}}. \tag{4.6}$$

As we will see in the following, there are many more such properties, but this list suffices for now.

From the lattice, we define the $\mathcal{W}$-extended fusion product $\hat{\otimes}$ by

$$\{2(2,1)_{\mathcal{W}} \oplus 2(4,1)_{\mathcal{W}}\} \,\hat{\otimes}\, (A)_{\mathcal{W}} := \lim_{n\to\infty} \left(\frac{1}{2n}\right)^3 (4n,1)^{\otimes 3} \otimes (A)_{\mathcal{W}} \tag{4.7}$$

$\hat{\otimes}$	$(2,1)_{\mathcal{W}}$	$(4,1)_{\mathcal{W}}$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$
$(2,1)_{\mathcal{W}}$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$2\mathcal{A}_2$	$2\mathcal{A}_2$
$(4,1)_{\mathcal{W}}$	$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$2\mathcal{A}_2$	$2\mathcal{A}_2$
$(\mathcal{R}_{2,1}^{1,0})_{\mathcal{W}}$	$2\mathcal{A}_2$	$2\mathcal{A}_2$	$2\mathcal{A}_{\mathcal{R}}$	$2\mathcal{A}_{\mathcal{R}}$
$(\mathcal{R}_{4,1}^{1,0})_{\mathcal{W}}$	$2\mathcal{A}_2$	$2\mathcal{A}_2$	$2\mathcal{A}_{\mathcal{R}}$	$2\mathcal{A}_{\mathcal{R}}$

Figure 10. Cayley table of the purely horizontal fusion algebra.

and obtain the fusions given in figure 10 where

$$\mathcal{A}_2 = (2,1)_{\mathcal{W}} \oplus (4,1)_{\mathcal{W}}, \qquad \mathcal{A}_{\mathcal{R}} = \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}}. \tag{4.8}$$

To appreciate this, we consider the two cases $A = (2,1)$ and $A = (4,1)$ and find

$$\begin{aligned}\{(2,1)_{\mathcal{W}} \oplus (4,1)_{\mathcal{W}}\} \,\hat{\otimes}\, (2,1)_{\mathcal{W}} &= \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2n}\right)^3 (4n,1)^{\otimes 3} \otimes (2,1)_{\mathcal{W}} \\ &= \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2n}\right)^2 (4n,1)^{\otimes 2} \otimes \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}} \\ &= \lim_{n\to\infty}\left(\frac{1}{2n}\right)(4n,1) \otimes \{(2,1)_{\mathcal{W}} \oplus (4,1)_{\mathcal{W}}\} \\ &= \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}}\end{aligned} \tag{4.9}$$

and likewise

$$\{(2,1)_{\mathcal{W}} \oplus (4,1)_{\mathcal{W}}\} \,\hat{\otimes}\, (4,1)_{\mathcal{W}} = \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}}. \tag{4.10}$$

We are still faced with the task of disentangling these results since the identification of the individual fusions such as $(2,1)_{\mathcal{W}} \,\hat{\otimes}\, (2,1)_{\mathcal{W}}$ is ambiguous at this point. However, since

$$(4k-2,1) \otimes (4k'-2,1) = \bigoplus_{j=|k-k'|+1}^{k+k'-1} \mathcal{R}_{4j-2,1}^{1,0} \tag{4.11}$$

and with the Virasoro decomposition of $(2,1)_{\mathcal{W}}$ in (4.3) in mind, it follows that the Virasoro decomposition of the fusion $(2,1)_{\mathcal{W}} \,\hat{\otimes}\, (2,1)_{\mathcal{W}}$ only involves rank-2 representations of the form $\big(\mathcal{R}_{4j-2,1}^{1,0}\big)_{\mathcal{W}}$. Initially comparing this with (4.9) and subsequently with (4.10), we conclude that

$$(2,1)_{\mathcal{W}} \,\hat{\otimes}\, (2,1)_{\mathcal{W}} = (4,1)_{\mathcal{W}} \,\hat{\otimes}\, (4,1)_{\mathcal{W}} = \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}}, \qquad (2,1)_{\mathcal{W}} \,\hat{\otimes}\, (4,1)_{\mathcal{W}} = \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}}. \tag{4.12}$$

In order to complete the Cayley table in figure 10, we also need to evaluate fusions like

$$(4,1)_{\mathcal{W}} \,\hat{\otimes}\, \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} = (2,1) \otimes (4,1)_{\mathcal{W}} \,\hat{\otimes}\, (4,1)_{\mathcal{W}} = (2,1) \otimes \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} = 2(2,1)_{\mathcal{W}} \oplus 2(4,1)_{\mathcal{W}} \tag{4.13}$$

and

$$\big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \,\hat{\otimes}\, \big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}} = (2,1) \otimes \big((4,1)_{\mathcal{W}} \,\hat{\otimes}\, \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}}\big) = 2\big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{4,1}^{1,0}\big)_{\mathcal{W}}. \tag{4.14}$$

The remaining fusions follow similarly.

Additional representations are obtained by fusing those above by the simple vertical (Virasoro-indecomposable) Kac representations $(1,2)$ and $(1,3)$. We thus define the rank-1 representations

$$(2\kappa, s)_{\mathcal{W}} := (2\kappa, 1)_{\mathcal{W}} \otimes (1, s) = \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)(2(2k-2+\kappa), s), \qquad s \in \mathbb{Z}_{2,3} \tag{4.15}$$

and the rank-2 representations

$$\left(\mathcal{R}_{2\kappa,s}^{1,0}\right)_{\mathcal{W}} := \left(\mathcal{R}_{2\kappa,1}^{1,0}\right)_{\mathcal{W}} \otimes (1, s) = \bigoplus_{k\in\mathbb{N}} (2k-2+\kappa)\mathcal{R}_{2(2k-2+\kappa),s}^{1,0}, \qquad s \in \mathbb{Z}_{2,3}. \tag{4.16}$$

Having ventured into the bulk part of the Kac table, we note the stability properties

$$\begin{aligned}
&(2\kappa, 1)_{\mathcal{W}} \otimes (1, 6n-3) = (2n-1)\big(2\kappa, 3\big)_{\mathcal{W}}, &&(2\kappa, 1)_{\mathcal{W}} \otimes (1, 6n) = 2n(2(3-\kappa), 3)_{\mathcal{W}} \\
&\left(\mathcal{R}_{2\kappa,1}^{1,0}\right)_{\mathcal{W}} \otimes (1, 6n-3) = (2n-1)\left(\mathcal{R}_{2\kappa,3}^{1,0}\right)_{\mathcal{W}}, &&\left(\mathcal{R}_{2\kappa,1}^{1,0}\right)_{\mathcal{W}} \otimes (1, 6n) = 2n\left(\mathcal{R}_{2(3-\kappa),3}^{1,0}\right)_{\mathcal{W}}.
\end{aligned} \tag{4.17}$$

4.2. Vertical component

The vertical component is developed and described in much the same way as the horizontal component above.

From the lattice, we choose to consider

$$\begin{aligned}
\lim_{n\to\infty} (1, 6n-3)^{\otimes 3} &= \bigoplus_{k\in\mathbb{N}} (2k-1)(2k-1, 1) \\
&= 3\left(\bigoplus_{k\in\mathbb{N}} (2k-1)(1, 6k-3)\right) \oplus 2\left(\bigoplus_{k\in\mathbb{N}} 2k\mathcal{R}_{1,6k}^{0,1}\right) \oplus \left(\bigoplus_{k\in\mathbb{N}} (2k-1)\mathcal{R}_{1,6k-3}^{0,2}\right).
\end{aligned} \tag{4.18}$$

Care has to be taken when disentangling this result in order to identify the $\mathcal{W}$-extended representations involved. First, we observe that the conformal weights of the Virasoro representations in the first sum all have rational part $1/3$ while the Virasoro representations in the second and third sums all have integer conformal weights. This allows us to separate the first sum from the other two and we have

$$(1, 3)_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} (2k-1)(1, 6k-3). \tag{4.19}$$

Now, fusing this with $(1, 3)$ gives

$$(1, 3)_{\mathcal{W}} \otimes (1, 3) = \bigoplus_{k\in\mathbb{N}} (2k-1)\big((1, 6k-3) \oplus \mathcal{R}_{1,6k-3}^{0,2}\big). \tag{4.20}$$

Having separated $(1, 3)_{\mathcal{W}}$ from this, we naturally identify the remaining part of the sum as the $\mathcal{W}$-extended rank-2 representation

$$\left(\mathcal{R}_{1,3}^{0,2}\right)_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} (2k-1)\mathcal{R}_{1,6k-3}^{0,2}. \tag{4.21}$$

The second sum in (4.18) can now be isolated and is identified as the $\mathcal{W}$-extended rank-2 representation

$$\left(\mathcal{R}_{1,6}^{0,1}\right)_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}} 2k\mathcal{R}_{1,6k}^{0,1}. \tag{4.22}$$

We thus assert that the limit of the triple fusion in (4.18) corresponds to the following sum of six $\mathcal{W}$-indecomposable representations:

$$3(1,3)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}} := \lim_{n\to\infty}(1,6n-3)^{\otimes 3}. \tag{4.23}$$

Here we emphasize a difference between the horizontal and vertical components. In the horizontal case, we could perform the disentanglement in (4.2) explicitly from the lattice by choosing the set of link states appropriately. As already indicated in the discussion following (4.2) and (4.3), this is not necessarily possible when non-trivial Jordan cells are present. One is faced with similar but more transparent complications in the Virasoro picture as well where the indecomposable rank-2 representations $\mathcal{R}_{1,3k}^{0,2}$ cannot be constructed individually from the lattice but only in combination with the Kac representations $(1,3k)$. To illustrate this, let us consider

$$(1,3)\otimes(1,3) = (1,3)\oplus\mathcal{R}_{1,3}^{0,2}, \qquad \chi\big[\mathcal{R}_{1,3}^{0,2}\big](q) = \chi_{1,1}(q)+\chi_{1,5}(q). \tag{4.24}$$

The Kac representations $(1,1)$, $(1,3)$ and $(1,5)$ are constructed by allowing exactly 0, 2 or 4 defects, respectively, to propagate through the bulk of the lattice, while the fusion $(1,3)\otimes(1,3)$ is evaluated by allowing 0, 2 or 4 defects to propagate through the bulk of the lattice. In the latter case, pairs of defects can be annihilated thus yielding a block-*triangular* matrix realization of the transfer fusion matrix. This block-triangularity may give rise to non-trivial Jordan cells as it does in the fusion $(1,3)\otimes(1,3)$. With reference to (4.24), it is now tempting to regard the indecomposable representation $\mathcal{R}_{1,3}^{0,2}$ as the result of allowing 0 or 4 defects to propagate through the bulk. Since defects could be annihilated in quadruples, this would indeed give rise to a block-triangular matrix. However, it turns out that no non-trivial Jordan cells are formed in this case implying that this choice of boundary condition simply corresponds to the *direct* sum of the two indecomposable rank-1 representations $(1,1)$ and $(1,5)$. As already mentioned, this phenomenon carries over to the $\mathcal{W}$-extended picture where the limiting process, though, obscures the clarity of the Virasoro example just discussed.

To continue, we could apply the analysis based on (4.18) to the infinite limit of the triple fusion of $(1,6n)$ with itself. Alternatively, we simply define the $\mathcal{W}$-extended rank-2 representation

$$\big(\mathcal{R}_{1,3}^{0,1}\big)_{\mathcal{W}} := (1,3)_{\mathcal{W}}\otimes(1,2) = \bigoplus_{k\in\mathbb{N}}(2k-1)\mathcal{R}_{1,6k-3}^{0,1} \tag{4.25}$$

and disentangle the fusions

$$\begin{aligned}\big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}}\otimes(1,2) &= 2\Bigg(\bigoplus_{k\in\mathbb{N}}2k(1,6k)\Bigg)\oplus\Bigg(\bigoplus_{k\in\mathbb{N}}(2k-1)\mathcal{R}_{1,6k-3}^{0,1}\Bigg)\\ \big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}}\otimes(1,2) &= 2\Bigg(\bigoplus_{k\in\mathbb{N}}2k(1,6k)\Bigg)\oplus\Bigg(\bigoplus_{k\in\mathbb{N}}2k\mathcal{R}_{1,6k}^{0,2}\Bigg)\end{aligned} \tag{4.26}$$

to identify

$$(1,6)_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}}2k(1,6k) \tag{4.27}$$

and subsequently

$$\big(\mathcal{R}_{1,6}^{0,2}\big)_{\mathcal{W}} = \bigoplus_{k\in\mathbb{N}}2k\mathcal{R}_{1,6k}^{0,2}. \tag{4.28}$$

We thus have the stability properties

$$\big(\mathcal{R}_{1,3\kappa}^{0,b}\big)_{\mathcal{W}}\otimes(1,2) = 2(1,3\kappa\cdot b)_{\mathcal{W}}\oplus\big(\mathcal{R}_{1,3\kappa}^{0,3-b}\big)_{\mathcal{W}} \tag{4.29}$$

$\hat{\otimes}$	$(1,3)_{\mathcal{W}}$	$(1,6)_{\mathcal{W}}$	$(\mathcal{R}_{1,6}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{1,3}^{0,2})_{\mathcal{W}}$	$(\mathcal{R}_{1,3}^{0,1})_{\mathcal{W}}$	$(\mathcal{R}_{1,6}^{0,2})_{\mathcal{W}}$
$(1,3)_{\mathcal{W}}$	$\mathcal{A}_3$	$\mathcal{A}_6$	$2\mathcal{B}_3$	$2\mathcal{B}_3$	$2\mathcal{B}_6$	$2\mathcal{B}_6$
$(1,6)_{\mathcal{W}}$	$\mathcal{A}_6$	$\mathcal{A}_3$	$2\mathcal{B}_6$	$2\mathcal{B}_6$	$2\mathcal{B}_3$	$2\mathcal{B}_3$
$(\mathcal{R}_{1,6}^{0,1})_{\mathcal{W}}$	$2\mathcal{B}_3$	$2\mathcal{B}_6$	$2\mathcal{A}_3\oplus 2\mathcal{B}_3$	$2\mathcal{A}_3\oplus 2\mathcal{B}_3$	$2\mathcal{A}_6\oplus 2\mathcal{B}_6$	$2\mathcal{A}_6\oplus 2\mathcal{B}_6$
$(\mathcal{R}_{1,3}^{0,2})_{\mathcal{W}}$	$2\mathcal{B}_3$	$2\mathcal{B}_6$	$2\mathcal{A}_3\oplus 2\mathcal{B}_3$	$2\mathcal{A}_3\oplus 2\mathcal{B}_3$	$2\mathcal{A}_6\oplus 2\mathcal{B}_6$	$2\mathcal{A}_6\oplus 2\mathcal{B}_6$
$(\mathcal{R}_{1,3}^{0,1})_{\mathcal{W}}$	$2\mathcal{B}_6$	$2\mathcal{B}_3$	$2\mathcal{A}_6\oplus 2\mathcal{B}_6$	$2\mathcal{A}_6\oplus 2\mathcal{B}_6$	$2\mathcal{A}_3\oplus 2\mathcal{B}_3$	$2\mathcal{A}_3\oplus 2\mathcal{B}_3$
$(\mathcal{R}_{1,6}^{0,2})_{\mathcal{W}}$	$2\mathcal{B}_6$	$2\mathcal{B}_3$	$2\mathcal{A}_6\oplus 2\mathcal{B}_6$	$2\mathcal{A}_6\oplus 2\mathcal{B}_6$	$2\mathcal{A}_3\oplus 2\mathcal{B}_3$	$2\mathcal{A}_3\oplus 2\mathcal{B}_3$

Figure 11. Cayley table of the purely vertical fusion algebra.

with further stability properties reading

$$\begin{aligned}(1,3\kappa)_{\mathcal{W}}\otimes(1,6n-3)&=(2n-1)\big\{(1,3\kappa)_{\mathcal{W}}\oplus\big(\mathcal{R}_{1,3\kappa}^{0,2}\big)_{\mathcal{W}}\big\}\\ \big(\mathcal{R}_{1,3\kappa}^{0,b}\big)_{\mathcal{W}}\otimes(1,6n-3)&=2(2n-1)\big\{(1,3(3-\kappa\cdot b))_{\mathcal{W}}\oplus\big(\mathcal{R}_{1,3\kappa\cdot b}^{0,1}\big)_{\mathcal{W}}\big\}.\end{aligned}\qquad(4.30)$$

In accordance with horizontal fusion, we use

$$\begin{aligned}\big\{3(1,3)_{\mathcal{W}}\oplus 2\big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}}\oplus\big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}}\big\}\,\hat{\otimes}\,(A)_{\mathcal{W}}&=\lim_{n\to\infty}\left(\frac{1}{2n}\right)^3(1,6n-3)^{\otimes 3}\otimes(A)_{\mathcal{W}}\\ &=\lim_{n\to\infty}\left(\frac{1}{2n-1}\right)^3(1,6n-3)^{\otimes 3}\otimes(A)_{\mathcal{W}},\end{aligned}\qquad(4.31)$$

when evaluating vertical fusions of $\mathcal{W}$-representations. With the abbreviations

$$\mathcal{A}_{3\kappa}=(1,3\kappa)_{\mathcal{W}}\oplus\big(\mathcal{R}_{1,3\kappa}^{0,2}\big)_{\mathcal{W}},\qquad \mathcal{B}_{3\kappa}=(1,3\kappa)_{\mathcal{W}}\oplus\big(\mathcal{R}_{1,3(3-\kappa)}^{0,1}\big)_{\mathcal{W}}\qquad(4.32)$$

and in much the same way as for the horizontal component, this yields the fusion rules in figure 11. Further following the analysis of the horizontal component, we introduce the rank-1 representations

$$(2,3\kappa)_{\mathcal{W}}:=(2,1)\otimes(1,3\kappa)_{\mathcal{W}}=\bigoplus_{k\in\mathbb{N}}(2k-2+\kappa)(2,3(2k-2+\kappa)).\qquad(4.33)$$

As required by consistency of notation, the representation $(2,3)_{\mathcal{W}}$ defined in (4.15) must agree with this expression, and indeed it does since

$$\bigoplus_{k\in\mathbb{N}}(2k-1)(4k-2,3)=\bigoplus_{k\in\mathbb{N}}(2k-1)(2,6k-3).\qquad(4.34)$$

It is likewise noted that

$$(2,6)_{\mathcal{W}}\equiv(4,3)_{\mathcal{W}},\qquad(4.35)$$

since

$$\bigoplus_{k\in\mathbb{N}}2k(2,6k)=\bigoplus_{k\in\mathbb{N}}2k(4k,3).\qquad(4.36)$$

We also introduce the rank-2 representations

$$\big(\mathcal{R}_{2,3\kappa}^{0,b}\big)_{\mathcal{W}}:=(2,1)\otimes\big(\mathcal{R}_{1,3\kappa}^{0,b}\big)_{\mathcal{W}}=\bigoplus_{k\in\mathbb{N}}(2k-2+\kappa)\mathcal{R}_{2,3(2k-2+\kappa)}^{0,b}.\qquad(4.37)$$

4.3. Combination of the two components

Our notation implies that

$$A \otimes (B)_{\mathcal{W}} \propto (A)_{\mathcal{W}} \otimes B. \tag{4.38}$$

Particularly, useful such relations are part of the stability properties

$$\begin{aligned}
&(2,1)_{\mathcal{W}} \otimes (1,3) = (2,3)_{\mathcal{W}} = (2,1) \otimes (1,3)_{\mathcal{W}} \\
&(4,1)_{\mathcal{W}} \otimes (1,3) = (4,3)_{\mathcal{W}} = \tfrac{1}{2}(4,1) \otimes (1,3)_{\mathcal{W}} \\
&\tfrac{1}{2}(2,1)_{\mathcal{W}} \otimes (1,6) = (2,6)_{\mathcal{W}} = (2,1) \otimes (1,6)_{\mathcal{W}} \\
&\tfrac{1}{2}(4,1)_{\mathcal{W}} \otimes (1,6) = (2,3)_{\mathcal{W}} = \tfrac{1}{2}(4,1) \otimes (1,6)_{\mathcal{W}}
\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
&(2,1)_{\mathcal{W}} \otimes \mathcal{R}_{1,3}^{0,b} = \big(\mathcal{R}_{2,3}^{0,b}\big)_{\mathcal{W}} = (2,1) \otimes \big(\mathcal{R}_{1,3}^{0,b}\big)_{\mathcal{W}} \\
&(4,1)_{\mathcal{W}} \otimes \mathcal{R}_{1,3}^{0,b} = \big(\mathcal{R}_{2,6}^{0,b}\big)_{\mathcal{W}} = \tfrac{1}{2}(4,1) \otimes \big(\mathcal{R}_{1,3}^{0,b}\big)_{\mathcal{W}} \\
&\tfrac{1}{2}(2,1)_{\mathcal{W}} \otimes \mathcal{R}_{1,6}^{0,b} = \big(\mathcal{R}_{2,6}^{0,b}\big)_{\mathcal{W}} = (2,1) \otimes \big(\mathcal{R}_{1,6}^{0,b}\big)_{\mathcal{W}} \\
&\tfrac{1}{2}(4,1)_{\mathcal{W}} \otimes \mathcal{R}_{1,6}^{0,b} = \big(\mathcal{R}_{2,3}^{0,b}\big)_{\mathcal{W}} = \tfrac{1}{2}(4,1) \otimes \big(\mathcal{R}_{1,6}^{0,b}\big)_{\mathcal{W}}.
\end{aligned} \tag{4.40}$$

To illustrate the derivation of these, we assume (4.39) when considering the first equality in the third line in (4.40),

$$\begin{aligned}
(2,1)_{\mathcal{W}} \otimes \mathcal{R}_{1,6}^{0,1} &= (2,1)_{\mathcal{W}} \otimes (1,6) \otimes (1,2) = 2(2,1) \otimes (1,6)_{\mathcal{W}} \otimes (1,2) \\
&= 2(2,1) \otimes \big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}} = 2\big(\mathcal{R}_{2,6}^{0,1}\big)_{\mathcal{W}}.
\end{aligned} \tag{4.41}$$

The $\mathcal{W}$-indecomposable rank-3 representations can be defined by

$$\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}} := \big(\mathcal{R}_{2\kappa,1}^{1,0}\big)_{\mathcal{W}} \otimes \mathcal{R}_{1,3}^{0,b} \tag{4.42}$$

or equivalently through

$$\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}} = \big(\mathcal{R}_{2,3\kappa}^{1,b}\big)_{\mathcal{W}} = \mathcal{R}_{2,1}^{1,0} \otimes \big(\mathcal{R}_{1,3\kappa}^{0,b}\big)_{\mathcal{W}}. \tag{4.43}$$

They have the stability properties

$$\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}} \otimes (1,2) = 2\big(\mathcal{R}_{2\kappa\cdot b,3}^{1,0}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{2\kappa,3}^{1,3-b}\big)_{\mathcal{W}}. \tag{4.44}$$

To get started with the evaluation of combined fusions, we consider

$$\begin{aligned}
(2,1)_{\mathcal{W}} \hat{\otimes} \big\{3(1,3)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}}\big\} &= \lim_{n\to\infty} \left(\frac{1}{2n-1}\right)^3 (2,1)_{\mathcal{W}} \otimes (1,6n-3)^{\otimes 3} \\
&= \lim_{n\to\infty} \left(\frac{1}{2n-1}\right)^2 (2,3)_{\mathcal{W}} \otimes (1,6n-3)^{\otimes 2} \\
&= \lim_{n\to\infty} \left(\frac{1}{2n-1}\right)^2 (2,1)_{\mathcal{W}} \otimes (1,6n-3)^{\otimes 2} \otimes (1,3) \\
&= (2,1)_{\mathcal{W}} \otimes (1,3)^{\otimes 3} = (2,1)_{\mathcal{W}} \otimes \left\{3(1,3) \oplus \frac{1}{2} 2\mathcal{R}_{1,6}^{0,1} \oplus \mathcal{R}_{1,3}^{0,2}\right\} \\
&= 3(2,3)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{2,6}^{0,1}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{2,3}^{0,2}\big)_{\mathcal{W}}.
\end{aligned} \tag{4.45}$$

Since the multiplicities appearing in the decomposition of the fusion $(2,1)_{\mathcal{W}} \hat{\otimes} \{3(1,3)_{\mathcal{W}}\}$ must be divisible by 3, we find that

$$(2,1)_{\mathcal{W}} \hat{\otimes} (1,3)_{\mathcal{W}} = (2,3)_{\mathcal{W}}. \tag{4.46}$$

Using a similar argument, we then deduce that

$$(2,1)_{\mathcal{W}} \hat{\otimes} \big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}} = \big(\mathcal{R}_{2,6}^{0,1}\big)_{\mathcal{W}} \tag{4.47}$$

and finally

$$(2,1)_{\mathcal{W}} \hat{\otimes} \big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}} = \big(\mathcal{R}_{2,3}^{0,2}\big)_{\mathcal{W}}. \tag{4.48}$$

We subsequently find

$$(2,1)_{\mathcal{W}} \hat{\otimes} \big(\mathcal{R}_{1,3}^{0,1}\big)_{\mathcal{W}} = (2,1)_{\mathcal{W}} \hat{\otimes} (1,3)_{\mathcal{W}} \otimes (1,2) = (2,3)_{\mathcal{W}} \otimes (1,2) = \big(\mathcal{R}_{2,3}^{0,1}\big)_{\mathcal{W}} \tag{4.49}$$

and hence

$$\begin{aligned}(2,1)_{\mathcal{W}} \hat{\otimes} (1,6)_{\mathcal{W}} &= \tfrac{1}{2}(2,1)_{\mathcal{W}} \hat{\otimes} \big\{\big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}} \otimes (1,2) \ominus \big(\mathcal{R}_{1,3}^{0,1}\big)_{\mathcal{W}}\big\} \\ &= \tfrac{1}{2}\big(\mathcal{R}_{2,3}^{0,2}\big)_{\mathcal{W}} \otimes (1,2) \ominus \tfrac{1}{2}\big(\mathcal{R}_{2,3}^{0,1}\big)_{\mathcal{W}} \\ &= \tfrac{1}{2}(2,1) \otimes \big\{2(1,6)_{\mathcal{W}} \oplus \big(\mathcal{R}_{1,3}^{0,1}\big)_{\mathcal{W}}\big\} \ominus \tfrac{1}{2}\big(\mathcal{R}_{2,3}^{0,1}\big)_{\mathcal{W}} \\ &= (4,3)_{\mathcal{W}}. \end{aligned} \tag{4.50}$$

The remaining fusions follow similarly or by simple applications of commutativity and associativity. Indeed, in our final example, we assume that all fusions but the ones between two rank-3 representations have been examined. Thus using commutativity, associativity and the fusion rules appearing in figures 5 through 8, we consider the fusion

$$\begin{aligned}\big(\mathcal{R}_{2,3}^{1,1}\big)_{\mathcal{W}} \hat{\otimes} \big(\mathcal{R}_{4,3}^{1,2}\big)_{\mathcal{W}} &= \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \hat{\otimes} \big(\mathcal{R}_{1,3}^{0,1}\big)_{\mathcal{W}} \hat{\otimes} \big(\mathcal{R}_{4,3}^{1,2}\big)_{\mathcal{W}} = \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \hat{\otimes} \big\{\mathcal{D}_{4,3}^{1,1} \oplus \mathcal{D}_{2,3}^{1,2}\big\} \\ &= \big(\mathcal{R}_{2,1}^{1,0}\big)_{\mathcal{W}} \hat{\otimes} \big\{4\big(\mathcal{R}_{2,3}^{1,0}\big)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{4,3}^{1,1}\big)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{4,3}^{1,2}\big)_{\mathcal{W}}\big\} \\ &= 4\mathcal{C}_3^{1,0} \oplus 2\mathcal{C}^{1,1} \oplus 2\mathcal{C}^{1,2} \end{aligned} \tag{4.51}$$

which is recognized as $\hat{\mathcal{C}}^1$, cf figure 9.

Self-consistency of our fusion prescription requires that the evaluation of a given fusion product based on (4.2) must yield the same result as the evaluation of the same fusion product based on (4.18), when both methods are applicable. This can be verified explicitly and stems from the fact that the stability properties (4.39) and (4.40) ensure that

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{1}{2n}\right)^3 (4n,1)^{\otimes 3} \otimes \big\{3(1,3)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}}\big\} \\ &\quad = \{2(2,1)_{\mathcal{W}} \oplus 2(4,1)_{\mathcal{W}}\} \hat{\otimes} \big\{3(1,3)_{\mathcal{W}} \oplus 2\big(\mathcal{R}_{1,6}^{0,1}\big)_{\mathcal{W}} \oplus \big(\mathcal{R}_{1,3}^{0,2}\big)_{\mathcal{W}}\big\} \\ &\quad = \lim_{n\to\infty}\left(\frac{1}{2n-1}\right)^3 \{2(2,1)_{\mathcal{W}} \oplus 2(4,1)_{\mathcal{W}}\} \otimes (1,6n-3)^{\otimes 3}. \end{aligned} \tag{4.52}$$

4.4. *Fusion subalgebras*

It is noted that there are many fusion subalgebras. We have already encountered two of them, namely the horizontal and vertical fusion algebras whose Cayley tables are given in figures 10 and 11, respectively. A noteworthy six-dimensional fusion subalgebra is

$$\langle(\mathcal{E},1),(\mathcal{O},1),(1,\mathcal{E}),(1,\mathcal{O})\rangle = \langle(\mathcal{E},1),(\mathcal{O},1),(1,\mathcal{E}),(1,\mathcal{O}),(\mathcal{E},\mathcal{E}),(\mathcal{O},\mathcal{O})\rangle. \tag{4.53}$$

It is generated by the four $\mathcal{W}$-representations

$$(\mathcal{E},1) := \lim_{n\to\infty}(4n,1)^{\otimes 3} = 2\mathcal{A}_2, \qquad (\mathcal{O},1) := \tfrac{1}{2}(2,1)\otimes(\mathcal{E},1) = \mathcal{A}_{\mathcal{R}}, \tag{4.54}$$

where it is noted that $\lim_{n\to\infty}(4n-2,1)^{\otimes 3} = \lim_{n\to\infty}(4n,1)^{\otimes 3}$, and

$$(1,\mathcal{E}) := \lim_{n\to\infty}(1,6n)^{\otimes 3} = \mathcal{A}_6 \oplus 2\mathcal{B}_6, \qquad (1,\mathcal{O}) := \lim_{n\to\infty}(1,6n-3)^{\otimes 3} = \mathcal{A}_3 \oplus 2\mathcal{B}_3. \tag{4.55}$$

$\hat{\otimes}$	$(\mathcal{E},1)$	$(\mathcal{O},1)$	$(1,\mathcal{E})$	$(1,\mathcal{O})$	$(\mathcal{E},\mathcal{E})$	$(\mathcal{O},\mathcal{O})$
$(\mathcal{E},1)$	$8(\mathcal{O},1)$	$8(\mathcal{E},1)$	$(\mathcal{E},\mathcal{E})$	$(\mathcal{E},\mathcal{E})$	$8(\mathcal{O},\mathcal{O})$	$8(\mathcal{E},\mathcal{E})$
$(\mathcal{O},1)$	$8(\mathcal{E},1)$	$8(\mathcal{O},1)$	$(\mathcal{O},\mathcal{O})$	$(\mathcal{O},\mathcal{O})$	$8(\mathcal{E},\mathcal{E})$	$8(\mathcal{O},\mathcal{O})$
$(1,\mathcal{E})$	$(\mathcal{E},\mathcal{E})$	$(\mathcal{O},\mathcal{O})$	$27(1,\mathcal{O})$	$27(1,\mathcal{E})$	$27(\mathcal{E},\mathcal{E})$	$27(\mathcal{O},\mathcal{O})$
$(1,\mathcal{O})$	$(\mathcal{E},\mathcal{E})$	$(\mathcal{O},\mathcal{O})$	$27(1,\mathcal{E})$	$27(1,\mathcal{O})$	$27(\mathcal{E},\mathcal{E})$	$27(\mathcal{O},\mathcal{O})$
$(\mathcal{E},\mathcal{E})$	$8(\mathcal{O},\mathcal{O})$	$8(\mathcal{E},\mathcal{E})$	$27(\mathcal{E},\mathcal{E})$	$27(\mathcal{E},\mathcal{E})$	$216(\mathcal{O},\mathcal{O})$	$216(\mathcal{E},\mathcal{E})$
$(\mathcal{O},\mathcal{O})$	$8(\mathcal{E},\mathcal{E})$	$8(\mathcal{O},\mathcal{O})$	$27(\mathcal{O},\mathcal{O})$	$27(\mathcal{O},\mathcal{O})$	$216(\mathcal{E},\mathcal{E})$	$216(\mathcal{O},\mathcal{O})$

Figure 12. Cayley table of the $\mathcal{E}$, $\mathcal{O}$ fusion subalgebra.

The remaining two representations are defined by

$$\begin{aligned}(\mathcal{E},\mathcal{E}) &:= (\mathcal{E},1)\,\hat{\otimes}\,(1,\mathcal{E}) = \bigoplus_{\kappa\in\mathbb{Z}_{1,2},b\in\mathbb{Z}_{0,2}} (6-2b)\big(\mathcal{R}_{2,3\kappa}^{0,b}\big)_{\mathcal{W}} \\ (\mathcal{O},\mathcal{O}) &:= (\mathcal{O},1)\,\hat{\otimes}\,(1,\mathcal{O}) = \bigoplus_{\kappa\in\mathbb{Z}_{1,2},b\in\mathbb{Z}_{0,2}} (3-b)\big(\mathcal{R}_{2\kappa,3}^{1,b}\big)_{\mathcal{W}},\end{aligned} \tag{4.56}$$

where $\big(\mathcal{R}_{2,3\kappa}^{0,0}\big)_{\mathcal{W}} \equiv (2,3\kappa)_{\mathcal{W}}$, and are seen to arise also in the fusions

$$(\mathcal{E},1)\,\hat{\otimes}\,(1,\mathcal{O}) = (\mathcal{E},\mathcal{E}), \qquad (\mathcal{O},1)\,\hat{\otimes}\,(1,\mathcal{E}) = (\mathcal{O},\mathcal{O}). \tag{4.57}$$

The Cayley table of the complete fusion subalgebra (4.53) is given in figure 12. A virtue of this fusion subalgebra is that it does *not* rely on any disentangling procedure.

5. Discussion

Two-dimensional critical percolation, with central charge $c = 0$, is viewed as the member $\mathcal{LM}(2,3)$ of the infinite series of Yang–Baxter integrable logarithmic minimal models $\mathcal{LM}(p,p')$ [8]. As in the rational case [49], the Yang–Baxter integrable boundary conditions give insight into the conformal boundary conditions [50] in the continuum scaling limit as well as the fusion of their associated representations. This enabled us in [8] to construct integrable boundary conditions labelled by (r,s) and corresponding to so-called Kac representations with conformal weights in an infinitely extended Kac table (figure 1). Moreover, from the lattice implementation of fusion, we obtained [34] the closed fusion algebra generated by these Kac representations finding that indecomposable representations of ranks 1, 2 and 3 are generated by the fusion process. Although there is a countable infinity of representations, the ensuing fusion rules are quasi-rational in the sense of Nahm [51], that is, the fusion of any two representations decomposes into a finite sum of representations. This is the relevant picture in the case where the conformal algebra is the Virasoro algebra. Of course, there is no claim, in the context of this logarithmic CFT, that the representations generated in this picture exhaust all of the representations associated with conformal boundary conditions. This picture is in stark contrast to the context of rational CFTs where all representations decompose into direct sums of a finite number of irreducible representations.

In this paper, we have reconsidered critical percolation (or more precisely the $\mathcal{LM}(2, 3)$ lattice model) in the continuum scaling limit to expose its nature as a 'rational' logarithmic CFT with respect to the extended conformal algebra $\mathcal{W} = \mathcal{W}_{2,3}$ [46]. Under the extended symmetry, the infinity of Virasoro representations are reorganized into a finite number of $\mathcal{W}$-representations. Following the approach of [44], we construct new solutions of the boundary Yang–Baxter equation which, in a suitable limit, correspond to these representations. Specifically, with respect to a suitably defined $\mathcal{W}$-fusion, we find that the representation content of the ensuing closed fusion algebra is *finite* containing 26 $\mathcal{W}$-indecomposable representations with eight rank-1 representations, 14 rank-2 representations and four rank-3 representations. We have also identified their associated $\mathcal{W}$-extended characters which decompose as finite non-negative sums of 13 $\mathcal{W}$-irreducible characters. Implementation of fusion on the lattice has allowed us to read off the fusion rules governing the fusion algebra of the 26 representations and to construct an explicit Cayley table. The closure of these representations among themselves under fusion is remarkable confirmation of the proposed extended symmetry. The extension of the present work to general $\mathcal{WLM}(p, p')$ is discussed in [47].

A somewhat surprising feature of our closed $\mathcal{W}$-extended fusion algebra of $\mathcal{WLM}(2, 3)$ is that there appears to be no natural identity $\mathcal{I}_{\mathcal{W}}$ expressed in terms of the fundamental Virasoro fusion algebra and with respect to the fusion multiplication $\hat{\otimes}$. Since the Kac representation $(1, 1)$ is the identity of the fundamental fusion algebra itself, it may be tempting to include it in the spectrum and identify it with $\mathcal{I}_{\mathcal{W}}$. However, we have

$$\{2(2,1)_{\mathcal{W}} \oplus 2(4,1)_{\mathcal{W}}\} \,\hat{\otimes}\, \mathcal{I}_{\mathcal{W}} = \lim_{n\to\infty} \left(\frac{1}{2n}\right)^3 (4n,1)^{\otimes 3} \otimes (1,1) = 0 \qquad (5.1)$$

demonstrating that this simple extension fails. We find it natural, though, to expect that one can extend our fusion algebra of $\mathcal{WLM}(2, 3)$ by working with the *full* Virasoro fusion algebra. We hope to discuss this and re-address the identity question elsewhere.

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